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ABSTRACT

This mathematics curriculum guide is to serve as a model and resource to Iowa school personnel as they develop or refine mathematics curriculum. It is based on research about how students learn mathematics and on recommendations from recent reports and studies. The philosophy and eight goals are identified. The second section of the guide concerns the scope, sequence, and strategies for teaching mathematics. Overriding themes are discussed: problem solving; mental computation, estimation, and reasonableness of results; and the use of technology. Comments on teaching mathematics concepts and on each strand in kindergarten through grade 8 are provided. General comments, strategies for three stages of understanding, overriding themes, and needed materials are presented for problem solving, number/numeration/number theory, geometry, measurement, probability, and statistics, ratio/proportion/percent, whole number, computation, decimal fractions, fractions, and integers. The curriculum for grades 9-12 is presented. Descriptions and content topics are given for algebra, analytic geometry, descriptive statistics, discrete mathematics, fifth-year mathematics, general mathematics I and II, geometry, technical mathematics, and trigonometry. Finally, evaluation is discussed briefly. A position paper on basic mathematical skills is appended. (MNS)

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A Curriculum Coordinating Committee Report

ED278569

A Guide to Curriculum Development in Mathematics

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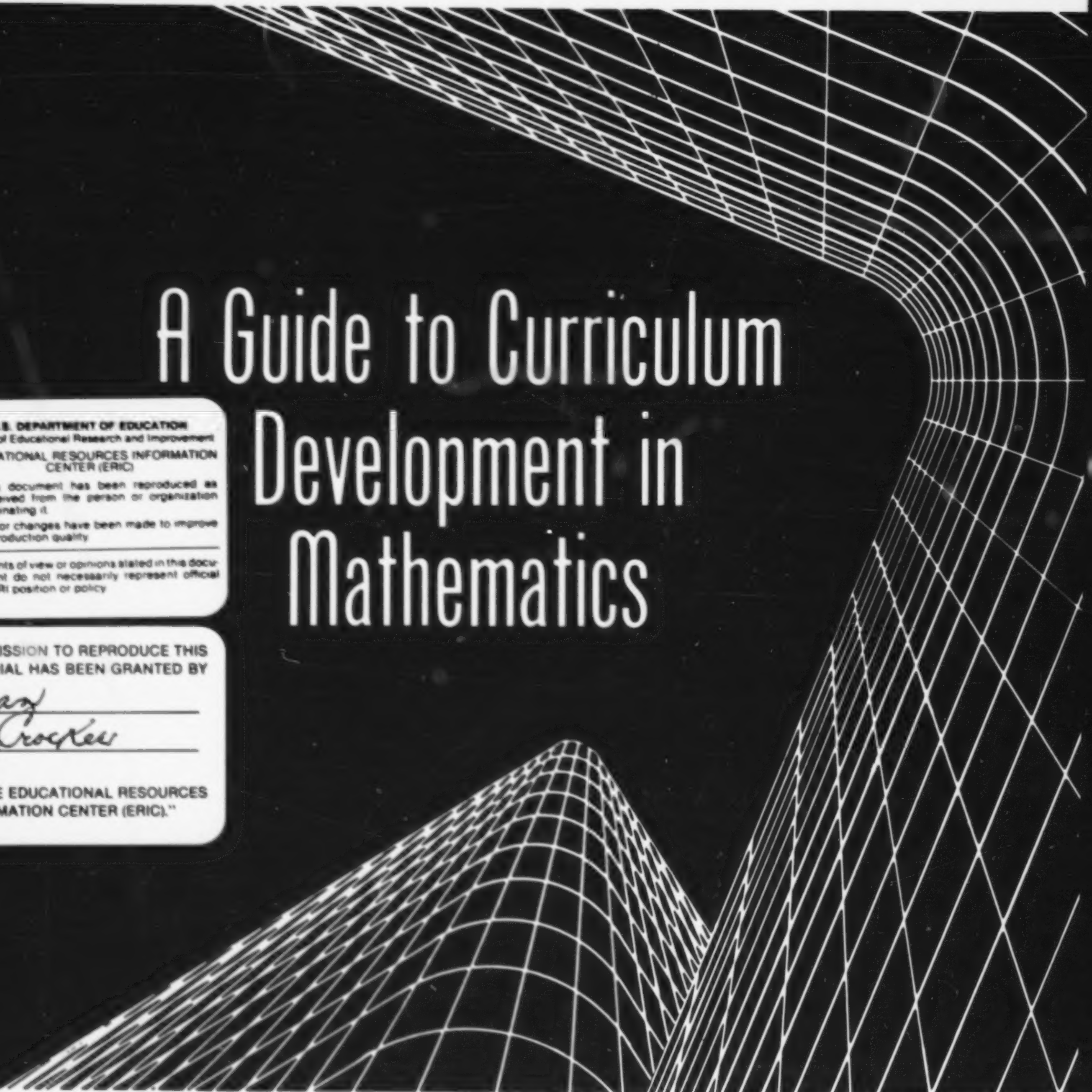

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Iowa Department of Education

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A GUIDE TO
CURRICULUM DEVELOPMENT
IN MATHEMATICS

Curriculum Coordinating Committee Report
Iowa Department of Education

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FOREWORD

Curriculum development is a synergistic, evolutionary, ongoing process which builds on successful practices and techniques. An effective curriculum stimulates each student to achieve his or her personal best and prepares the student to meet a rapidly changing world. An effective curriculum is a challenge to students and educators. It challenges students to continually achieve and educators to continually develop their teaching techniques.

This guide is a result of a 1985 mandate from the Iowa General Assembly, calling for the Department of Education to develop "subject matter committees and committees that cross subject matter lines for coordination of curriculum at all education levels."

This mandate was a response to one of the major recommendations of the Legislature's Excellence in Education Task Force report of 1984, *First in the Nation in Education (FINE)*. The Department of Education based its plan for implementing the legislation on recommendations from the report. The Department established six subject matter committees of 15 to 17 members for the arts, foreign languages, language arts, mathematics, sciences, and social studies. The committees were composed of elementary teachers, secondary teachers, merged area school faculty, college and university faculty members, area education agency consultants, school district curriculum coordinators, and other education professionals.

The committees devoted their first year of activity to the vertical articulation of each subject area. This guide is the result of that work. The FINE report notes the importance of horizontal articulation across subject areas, developing such skills as reading and comprehending, writing, speaking and listening, problem solving, conceptualizing, reasoning, analyzing, and learning to learn. This guide will be used as a basis for horizontal articulation in coming years.

The best curriculum is not mandated or enforced from outside. It results from the active involvement of teachers and administrators striving to meet the needs of students. It combines new processes and research about effective curriculum with the experience and expertise of teachers, administrators, and specialists.

This publication is designed to guide faculties and administrators in developing curriculum and improving instruction in mathematics. It is intended to help districts enhance and build upon their current local curriculum.

Robert D. Benton

Robert D. Benton, Ed.D.
Director, Iowa Department of Education

PREFACE

Developing a curriculum model for mathematics in kindergarten through grade 12 was a monumental task. It involved: (1) studying what is presently done in the Iowa curriculum; (2) consulting national reports to see what trends and recommendations are being stressed; (3) determining what should be retained; and (4) writing a guide that would infuse the recommendations from national reports with the content areas and strategies that provide appropriate experiences for students.

The committee wishes to emphasize that our work on the curriculum model should not be considered a completed task. This guide is a recommendation which we hope will be a valuable aid to local schools as they study and build their own mathematics curriculum. This is a working publication which will continue to evolve as the creative minds of the mathematics teachers of Iowa add, delete, and review materials to fit the needs of their own students and locale.

As committee members, we were selected because of our varied job responsibilities. The committee work provided a professional, learning experience for us. We hope that as you use this guide, you will grow professionally as we did.

The Mathematics Curriculum Coordinating Committee

ACKNOWLEDGEMENTS

We wish to thank our colleagues who gave of their time and expertise to contribute and react to various portions of this curriculum model.

As a committee, we wish to express our appreciation to Theresa Carter for her secretarial assistance, which was in addition to her normal responsibilities. She was always cheerful, pleasant, and cooperative as we worked to develop this curriculum model for mathematics.

Due to Department reorganization, Claudette Williams assumed the secretarial responsibilities as this guide was going through the final stages of editing. In assuming this task and becoming familiar with the format and contents of the guide, she was always pleasant and cooperative as she worked with the Publications staff members to prepare the final copy for printing.

Thus as a committee, we wish to thank both Theresa and Claudette and our colleagues for their contributions in the preparation of this mathematics curriculum model.

MATHEMATICS CURRICULUM COORDINATING COMMITTEE

This publication was developed with the aid and advice of a 15-member committee of mathematics professionals from around the state. As representatives of the readers and users of the model, they helped to ensure that this publication reflects the needs and views of those who will use it.

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Rationale

Philosophy

Goals

RATIONALE

The purpose of this mathematics curriculum guide is to serve as a model and resource to school personnel as they develop or refine their mathematics curriculum. This model is based on research about how students learn mathematics and on recommendations from recent reports and studies for educating students for the 21st century. Also, this model takes into consideration the future interests, aspirations, and needs of individual students.

This guide describes a curriculum for a changing society. It represents the combined efforts of the curriculum coordinating committee members by providing recommendations to administrators and faculty members as they look to the future to establish long- and short-range goals for the school's mathematics program. The intent of the recommendations is to provide direction and assistance to local school district personnel and patrons as they ascertain whether their mathematics curriculum and program reflect the trends and recommendations identified in reports and research. The desirable outcome for users of this guide would be to provide a better mathematics curriculum and instructional program for all students.

PHILOSOPHY

A high-quality mathematics program dares students to experience mathematics rather than observe it. Mathematics is for life and mathematics is a part of life. Students must be actively engaged in their learning.

Today's students must be provided with an understanding of mathematics. They must be provided with experiences that will cause them to become active, flexible thinkers and users of mathematics. It is critical that all students — regardless of race, national origin, sex, or disability — be involved in and see themselves reflected in the mathematics curriculum. It is essential that all students are engaged in a program that contains appropriate mathematical content and learn the content to form a knowledge base. The program must be one which can be expanded in the future to broaden their career and economic horizons and allow the students to adapt with the changing times.

The primary focus of the mathematics curriculum is to help students become good problem solvers. Learning experiences must cycle between using problems to motivate knowledge base development and using the knowledge base to solve problems. To accomplish this, problem solving should include the processing of information, thinking analytically, coping with changes, and making decisions by using mathematics with varying degrees of sophistication. Classroom instruction should provide for a natural learning sequence which allows for the transition from concrete to semi-concrete to semi-abstract to abstract learning experiences.

The instructional climate must allow students to communicate their mathematical ideas freely and to turn mathematical "errors" into positive learning experiences. Discussions should occur using the language of mathematics to verbalize the processes used to develop concepts and solve problems. Experiences should be designed to allow students to interact with each other and the teacher while attaining a knowledge base and solving problems.

Since students comprehend mathematics through various learning strategies, a variety of evaluation techniques must be employed. Evaluation of the students' knowledge base and problem solving ability must not be limited to only paper and pencil testing.

MATHEMATICS GOALS

Students will be provided experiences which:

1. emphasize problem solving and thinking skills.
2. give a broad perspective of the mathematics content structure, and the interrelationships among the various structural strands/branches.
3. consider different learning styles by using a variety of instructional strategies and materials — manipulative, visual, and print.
4. emphasize a participatory role for learning by using mathematical language, oral discussion, writing, listening skills, and observing skills.
5. create mutual respect and equal treatment regardless of sex, race, disability, or national origin.
6. expand career and economic horizons.
7. incorporate technology as a thinking and learning tool.
8. assess performance through a variety of evaluation techniques.

Scope
Sequence
Strategies

OVERRIDING THEMES FOR THE MATHEMATICS CURRICULUM

Problem solving, mental computation, estimation, and reasonableness of results are overriding themes identified by the Mathematics Curriculum Coordinating Committee for the entire curriculum model. Each should permeate the mathematics curriculum. The most important of these is problem solving.

PROBLEM SOLVING

Problem solving should be the focus of the mathematics curriculum. At every grade level and at every ability range, problem solving should be used to introduce new concepts and skills as well as to develop the students' understanding of and ability to solve a variety of problems.

This philosophy agrees with that of the National Council of Teachers of Mathematics (NCTM) and the National Council of Supervisors of Mathematics (NCSM). The NCTM states:

> **Problem Solving Must be the Focus of School Mathematics in the 1980's**
>
> The development of problem-solving ability should direct the efforts of mathematics educators through the next decade. Performance in problem solving will measure the effectiveness of our personal and national possession of mathematical competence.
>
> Problem solving encompasses a multitude of routine and commonplace as well as nonroutine functions considered to be essential to the day-to-day living of every citizen. But it must also prepare individuals to deal with the special problems they will face in their individual careers.
>
> Problem solving involves applying mathematics to the real world, serving the theory and practice of current and emerging sciences, and resolving issues that extend the frontiers of the mathematical sciences themselves.

> This recommendation should not be interpreted to mean that the mathematics to be taught is solely a function of the particular mathematics needed at a given time to solve a given problem. Structural unity and the interrelationships of the whole should not be sacrificed.
>
> True problem-solving power requires a wide repertoire of knowledge, not only of particular skills and concepts but also of the relationships among them and the fundamental principles that unify them. Each problem cannot be treated as an isolated example. This recommendation looks toward the need to solve problems in an uncertain future as well as here and now. As such, mathematics needs to be taught as mathematics, not as an adjunct to its fields of application. This demands a continuing attention to its internal cohesiveness and organizing principles as well as to its uses.[1]

The NCSM finds problem solving central to mathematics study:

> Learning to solve problems is the principal reason for studying mathematics. Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations. Solving word problems in texts is one form of problem solving, but students also should be faced with non-textbook problems. Problem-solving strategies involve posing questions, analyzing situations, translating results, illustrating results, drawing diagrams, and using trial and error. In solving problems, students need to be able to apply the rules of logic necessary to arrive at valid conclusions. They must be able to determine which facts are relevant. They should be unfearful of arriving at tentative conclusions and they must be willing to subject these conclusions to scrutiny.[2]

Specific strategies and ideas for incorporating problem solving into the mathematics program are listed within strands and as a strand itself in the K-8 section of the model.

MENTAL COMPUTATION, ESTIMATION, REASONABLENESS OF RESULTS

The abilities to compute mentally, to estimate, and to recognize the reasonableness of results to problems are important at all levels of mathematics. Although the advent of inexpensive calculators and other electronic computational devices in society have made arithmetic processes easier, students still need to be able to predict the answer to problems and to know whether or not the answer is reasonable.

The NCTM and NCSM have recommended ways to develop these skills. According to the NCTM, "Teachers should incorporate estimation activities into all areas of the program on a regular and sustaining basis, in particular encouraging the use of estimating skills to pose and select alternatives and to assess what a reasonable answer may be."[3]

The NCSM covers these areas in its position paper, Basic Mathematical Skills—Ten Basic Skill Areas:

> Estimation and Approximation
>
> Students should be able to carry out rapid approximate calculations by first rounding off numbers. They should acquire some simple techniques for estimating quantity, length, distance, weight, etc. It is also necessary to decide when a particular result is precise enough for the purpose at hand.
>
> Alertness to the Reasonableness of Results
>
> Due to arithmetic errors or other mistakes, results of mathematical work are sometimes wrong. Students should learn to inspect all results and to check for reasonableness in terms of the original problem. With the increase in the use of calculating devices in society, this skill is essential.
>
> Appropriate Computational Skills
>
> Students should gain facility with addition, subtraction, multiplication, and division with whole numbers and decimals. Today it must be recognized that long, complicated computations will usually be done with a calculator. Knowledge of single-digit number facts is essential and mental arithmetic is a valuable skill. Moreover, there are everyday situations which demand recognition of, and simple computation with, common fractions.[4]

A student's success at estimating is based on being proficient at the basic facts. Despite the allure of calculators to perform messy computations quickly, students must be encouraged to be proficient in the basic facts, place value, and mental arithmetic so they can easily calculate or give approximate answers to problems such as 500 divided by 2, 7.8 plus 3.1, and 88 multiplied by 3.

The ability to estimate increases as students are given more opportunities. Estimation should not be restricted to chapters in a book or be delayed until students begin computation activities. Since there is no specific age at which estimation activities should begin, appropriate activities must be introduced at the earliest age possible and continue throughout high school.

TECHNOLOGY

The use of technology as a tool has also been identified as an overriding theme for the total mathematics program. In the recommendations of this curriculum model, frequent references are made to the use of calculators or computers as tools. Each reference implies that the students must first know how to use the calculator or computer. Otherwise the technology becomes an obstacle, rather than the intended tool.

The importance of using the technology as a tool cannot be overstated. Uses of the technology for other purposes are far less valuable. Using a calculator to check answers to problems done with paper and pencil is not appropriate. Computer software must be carefully selected and be specific to the objectives. Using the computer as an electronic textbook often wastes computer power and student time.

In the recommendations for teaching primitive number concepts, frequent references are made to reading symbols aloud and recording symbols given orally. This must include reading and recording on the calculator or computer, as well as with paper and pencil.

Techniques in using a calculator should be taught from the earliest stages of learning so that the calculator can become a useful and familiar tool. As students grow in their mathematical competence, they should also grow in their competence with a calculator.

Proficiency with the calculator increases students' ability to apply mathematics to problems arising from real-life situations, as opposed to artificial problems, structured for "easy" numbers.

By fifth or sixth grade students should be competent with multi-step calculations on a calculator. Competence implies familiarity with order of operations and the use of the memory. Competence, however, cannot be assumed at any level. Teachers must always check to be sure that students can use the calculator appropriately before teaching concepts that depend on that skill.

Scientific calculators should be introduced whenever the four-function calculator becomes inadequate for the mathematics. Use of the more powerful tool should be taught parallel to the mathematics topics for which it is to be used.

The computer programming necessary for mathematical problem solving can be taught as students need it. A very limited subset of commands is sufficient for computer use in mathematical problem solving.

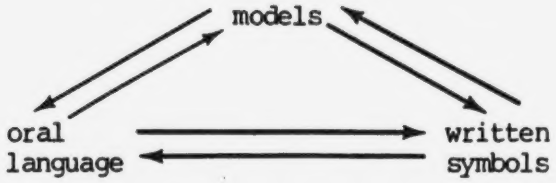
Teachers often find that students become very excited about the computer and want to know what else they can do with it. This tempts teachers to teach students all they know about the computer, at the expense of time needed to teach mathematics. This temptation must be resisted. Computer science and data processing should be taught in computer science or data processing courses, not in mathematics courses.

TEACHING MATHEMATICS CONCEPTS*

Concept learning involves several components. Included are (1) generating the idea, (2) recognizing instances of the idea, (3) representing the idea, (4) translating from one form of representation to another, and (5) learning the properties of the specific concept.

To help children generate the idea for a specific concept, teachers need to present examples of the concept and identify them by using the name of the concept. The language used at this stage not only helps children learn the name, but helps them focus on the particular attributes that are relevant characteristics in recognizing the instances of the concept. Soon after children can recognize instances of the concept, counterexamples and examples need to be mixed so that children are forced to focus on only the relevant attributes. Teachers need to supplement the textbook to provide sufficient experience for children to recognize examples of a new concept.

Next, children need to learn to represent the concept using models, oral language, and written symbols. Note that the use of models and oral language have already been presented to the students as they were generating the idea and learning to recognize instances of the concept. Research indicates that children who can represent a concept with models and oral language and very fluently translate from one type of representation to the other are better prepared to learn to use written symbols as a means of representing the concept. Many of the tasks children are asked to perform in elementary mathematics textbook exercises while they are initially learning the concept involve these translations. The six translations that children need to master are shown in the diagram below.



Children who work through textbooks without interaction with models and appropriate language are not learning all of the six translations. Teachers need to supplement the textbook to ensure that children can fluently deal with all the different types of representations for a concept and the translations among them. Otherwise, the student's application of the concept to real-world situations will be limited.

*Permission to use this unpublished paper "Teaching Mathematics Concepts" was given by the author, Edward C. Rathmell, University of Northern Iowa.

If there are different types of models that can be used to represent a concept, the children need to learn to use each different type. The purpose of studying these concepts is to be able to use them in the real world. That can not happen unless children recognize it. Since we cannot predict how the children will encounter this concept in real-world settings, we need to prepare them to recognize the concept in any form. To facilitate learning, children should become familiar with and develop competence with one model before another model is introduced. However, each of the different types of models needs to be included in the elementary curriculum at some stage.

To make effective instructional decisions, teachers need to know to what extent children understand a new concept. Teachers need to monitor the language and thinking children are using as well as their written work to get adequate feedback for making these instructional decisions.

Finally, after children have learned to recognize and represent a new concept and can easily make translations from one type of representation to the others, they need to begin learning the properties of that concept. They also need to learn which transformations can be applied to an instance of the concept without changing it and which transformations do change it. These ideas enable children to broaden their understanding of the concept and apply it in new and varied situations.

HOW TO INTERPRET THE INFORMATION IN THE K-8 STRANDS

The K-8 strands are formatted as follows:

Strand:	This is the name of the strand.
General Comments:	This is a narrative section that provides an overview of the basic philosophy behind teaching this strand and discusses some of the concerns related to it.
Strategies/Hints:	This section breaks down the strand by specific topics and presents suggested activities that students should be provided. The activities are identified as Stage I, Stage II, or Stage III. See "An Explanation of 'Strategies/Hints'" below.
Overriding Themes:	This section provides brief ideas about integrating the four overriding themes with the strand.
Instructional Materials:	This section is a list of possible materials that could be used to assist in teaching the strand.

AN EXPLANATION OF "STRATEGIES/HINTS"

Every classroom consists of students whose backgrounds and mental maturity are at a variety of levels. One third grade student may be at his or her highest level of understanding of two-digit addition with regrouping, using blocks and no symbols. Another child in the same class may understand two-digit addition with no regrouping and be doing so successfully at the symbolic level. Yet another child may be performing the computations symbolically for all addition but has no true understanding of the concept. Therefore, it is important to provide a wide variety of activities, from concrete to semi-concrete/semi-abstract to abstract. These activities are suggested as follows:

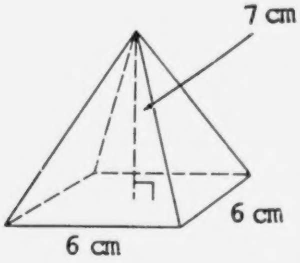
Stage I: Generally concrete, using objects/manipulatives to introduce the skill or concept or problem regardless of the age or grade of the learner or the degree of difficulty of the material.

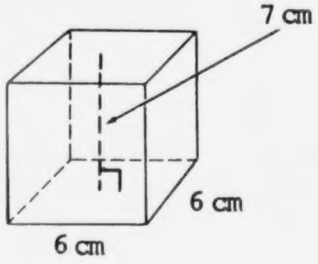
e.g. 1. When comparing the volume of a square-based pyramid to a square-based prism of equal length (l), width (w), and height (h) the students and teachers should use hollow objects and pour one container's contents into the other. No symbols or formulas are attached to the activity at this point.

2. To understand the sum of 3 and 2, the child physically combines three objects with two objects. Facts are stated orally, "Three and two make five" or "Three plus two is five."

Stage II: Activities at this level must link the objects in Stage I with pictures, drawings, or other visual representations such as tallies or dots with the corresponding symbol.

e.g. 1.





2.

///   //
3   2

/////
5

Stage III: Activities at this stage are at the abstract level, using symbols.

e.g. 1. Volume of prism is $V = l \cdot w \cdot h$

Volume of pyramid is $V = \frac{l \cdot w \cdot h}{3}$

2. $3 + 2 = 5$

Users of this guide need to realize that the three stages of learning identified for each of the K-8 strands should not be associated with grade level clusters. In other words, Stage I does not imply primary grades, Stage II does not imply intermediate grades, and Stage III does not imply middle school/junior high grades. When considering the recommended content in this guide, users should realize that a wide range of activities from concrete through abstract must be presented, developed, and revisited throughout the year. Teachers must not assume that a learner at the intermediate or advanced level of content does not need concrete or connecting level activities.

Teachers should not expect each child to move linearly from concrete to connecting to abstract activities for each strand. Learners at all levels need to move among the stages, sometimes cycling back to an earlier stage to reaffirm ideas or to gain new insights. Consideration must be given to the unique way that each learner's brain functions and develops. It also needs to be recognized that mathematics textbooks do not generally allocate many lessons or activities to the concrete, semi-concrete, or semi-abstract levels of learning.

Teachers who understand that learning is not a lock-step, mechanistic procedure that takes place as described in basal textbooks, take pleasure in providing students with their most effective means of learning. They witness joy and confidence emerge in their students as they claim the new knowledge as their own.

Strand: PROBLEM SOLVING

> Knowledge is always a personal possession, and it is always based on the knower's personal experience, whether that experience is direct or vicarious, concrete or abstract. We can and do acquire knowledge from others, but we only do so with lasting effect if we have somehow been able to make it our own.
> —John S. Mayher and Nancy B. Lester.[5]

General Comments:

Problem solving is a creative thinking process, essential throughout life. Problem solving requires opportunities to explore, talk, write, reflect, review, and revise. Students must also be allowed to be active pursuers of familiar and unique strategies to arrive at solutions for a wide of non-routine problems. This creative problem-solving process is best accomplished when students work in small groups, talk about the problem together as a team, and share their strategies for solutions with the whole class. Students need time to examine the structures of the problems, time to compare them to previously encountered situations, and time to determine whether to apply known strategies or to explore new ones. They must have time to internalize their learning through talking and writing about their thinking processes. Research has shown that students who are aware of their own thinking processes are more successful problem solvers. It is not uncommon for one problem to take one or more entire class periods.

The goal of mathematical problem solving is to give students the confidence to approach any new situation with an array of strategies and reasoning skills. Teachers and students, operating together as learners, extend problem-solving strategies and create new ones together through modeling, questioning, sharing, and refining. Students must not have teacher-directed strategies forced on them; rather, they must be given opportunities to assimilate the new information with their own knowledge and experiences when their mental development is ready. Repeating experiences (revisitations) of problems with similar structures, throughout the entire school year and throughout the child's schooling experience, will allow the gradual, easy development of the desired new strategies. When students begin to observe the structures of the problem, they can transfer strategies from one successfully solved problem to the new, related problem. As students observe and internalize other more efficient strategies that they learn from their classmates during the class team sharing period, they will begin to include them in their own repertoire for later use.

The emphasis of problem solving must be on process rather than on speed or final answers. "A problem is not necessarily 'solved' because the correct response has been made. A problem is not truly solved unless the learner understands what s/he has done and knows why her/his actions were appropriate. Instead of being 'protected' from error, the child should many times be exposed to error and be encouraged to detect and demonstrate what is wrong, and why." [7]

Problem solving must be an integral part and major focus of the mathematics program and not be taught as an isolated strand. Therefore, problem solving must begin in kindergarten and not be delayed until the students know the concept of number, algorithms, or are able to read.

A distinction must be made between the usual application word problems found in textbooks at the end of the computation sections and the nonroutine problem-solving focus of this strand:

> Application problems are generally exercises specific to the process studied and are designed for practice. This usually consists of students reading the problem, deciding how to re-write it as a sentence using mathematical symbols, and solving it. Checking the answer consists of plugging the answer back into the sentence and re-working for truth. The focus of application problems is on the algorithm rather on multiple strategies and processes.
>
> There is a need for solving application problems. However, it is more appropriate for them to be used as a vehicle for introducing new algorithms and concepts rather than to be used strictly as practice exercises.
>
> Nonroutine problems require creativity or originality. This type of problem solving refers to situations in which the students encounter a task for which there is no obvious or immediate step-by-step procedure that leads the students to a quick solution.

The background experiences of the students determine which situations are nonroutine problems.

The emphasis on similarities and differences of problems must focus on the structure of the problem.[8] Many textbook companies place the emphasis of problem similarity on content or on key-word strategies. Too often key words in mathematics have a double meaning because of the students' varied past experiences, leading children down the wrong path to solution as well as focusing them on the wrong aspect of problem solving. According to an NCTM yearbook, "Children can profit from some instruction directed toward key words, but this instruction should point out the dangers as well as the benefits of relying on key words."[9]

Teachers must be alerted to the recent textbook adaptations of the four-step Polya Model for solving problems: understand the problem, devise a plan, carry out the plan, and look back. In an effort to simplify the problem-solving process, many textbook publishers have not only removed the powerful, on-going questioning techniques essential to developing problem-solving skills, but have isolated the four steps, teaching them independently and in the stated order. Teachers and students will be much more successful in problem solving when teachers allow and encourage students to weave in and among the four steps, asking many questions of themselves and each other throughout the process. This approach allows the brain to function in its natural, non-linear mode and sets up the learning climate for students to take full advantage of the activity of thinking about the entire thinking process (metacognition) rather than about discrete steps.

The teacher's role in problem solving is different than the traditional role of dispenser of knowledge. The teacher becomes a facilitator and learner in a student-centered classroom. The teacher moves from group to group, becoming an active listener, learning with the students, questioning and extending their thinking while turning the act of teaching over to the students. The teacher must provide struggle time for both the students and the teacher to work with the problem, exploring all its facets and internalizing its structures. The most important role of the teacher is to provide a climate which develops in students the confidence to take risks and a tolerance for being wrong.

Strategies/Hints:

To develop an understanding of problem solving, students must have the opportunity to be involved in learning activities which emphasize:

o **STRATEGIES**

Stage I

A. Students at this level can learn and apply the following strategies using the actual object or physical representations of the object in the problem:

1. act it out
2. build a model
3. manipulate the model
4. make a simpler problem
5. look for/make patterns
6. work it backwards
7. guess and test

8. list all possibilities

9. logical deduction

10. tables

11. bar graph

B. Students will show they understand the problem by talking and writing about the process of solving the problem.

Stage II

A. In this level problem solving moves from concrete experiences to semi-concrete and to semi-abstract methods by using drawings, pictures, tallies, and other pictorial representations for the strategies listed at the Stage I Level. Additionally, students are able to:

1. make a sketch

2. make a line graph

3. make a pie graph

B. Students will validate their understanding of the problem by talking and writing about the strategy they used to solve the problem, why the strategy worked or did not work, and the discoveries they made while working. Learning logs (diaries) provide opportunities for students to look back for recurring themes in problem solving and to build on previous learning.

Stage III

A. The final level of problem solving moves from semi-concrete/semi-abstract to abstract methods of problem solving by writing and solving number sentences and/or formulas. Students will be able to use all previously learned problem-solving strategies abstractly. Advanced problem solvers should be encouraged to return to the concrete level, if necessary, for understanding before applying the more advanced strategies.

B. Students will validate their understanding of the problem by talking and writing about the process of solving the problem and why their strategy worked or did not work, and the discoveries they made while working.

Summary:

STAGE I	STAGE II		STAGE III	
@				act it out
@				build a model
	*	&		draw a model/picture, sketch
@	*	&	#	make a simpler problem
@				manipulate the model
@	*	&	#	look for/make patterns
@	*	&	#	work backwards
@	*	&	#	guess and test
@	*	&	#	exhaustive listing/list all possibilities
@	*	&	#	logical deduction
				represent the data:
@	*	&	#	tables
@	*	&	#	graph (depending—bar/line/pie)
			#	number sentence
			#	algebraic sentence/formula

Legend: @ concrete (objects)
* semi-concrete (pictures)
& semi-abstract (tallies, shapes)
# abstract (symbolic)

Strand: NUMBER, NUMERATION, NUMBER THEORY

General Comments:

Students come to school with enthusiasm and creativity, but they also come with a need to use concrete objects and receive instruction at their level of understanding. Rushing into the use of symbols (numerals) is detrimental to their natural development and may have a long-term adverse effect on their creativity and enthusiasm for learning mathematics. This is a perfect age to keep out mystery and keep in the magic of mathematics by providing activities compatible with their level of development.

Most children enter kindergarten with some number concepts. They often know oral names for numbers to at least 10. They can count small groups of objects and have had some experience with matching sets such as one sock for each shoe or one place setting for each family member. During the first half of kindergarten, these experiences should be reinforced and extended through active learning experiences that cannot be replaced with worksheets. Without dealing with the numerical symbols (numerals), students should have practice matching sets, orally counting, and solving verbal problems by counting. When numerals are introduced, the emphasis should be on selection based on recognition, rather than reproducing the symbols.

Beginning numeration activities should focus on recording the number of tens and ones rather than the oral names for two-digit numbers. The concept of tens and ones (and later hundreds, thousands, . . .) is crucial to understanding and effective use of the operation algorithms. Students need many opportunities to "trade" 10 units for one ten, 10 tens for one hundred, . . . Students also need to be able to translate from any one of three representations of a number (model, oral name, written numeral) to the other two representations.

Number theory is an area of mathematics that begins early, on an informal basis, when students do skip counting or shade even numbers on a hundreds chart. These and more complex number theory exercises help develop the broader area of intuitive "number sense" which enables apt mathematics students to develop short cuts in computation and make leaps in problem solving. "Number sense" is not a mechanized process to crank out an answer.

Strategies/Hints

To develop an understanding for the basic concepts of number, numeration, and number theory, students must be given an opportunity to do activities involving:

o NUMBER

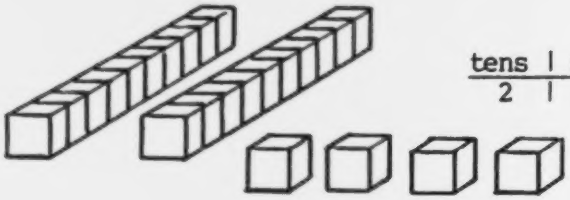
Stage I

A. Recognize that the rearrangement of objects does not change the number -- conservation of a number.

B. Match oral names and numerals with given sets and construct sets to match oral names and numerals. In the latter case students can make any number of arrangements using six toothpicks, or beans, counters, ...

C. Order sets of objects from fewest to most.

D. Write the corresponding digits representing number of tens and ones for a given arrangement of place value blocks and given the respective digits arrange the blocks accordingly.



tens	ones
2	4

E. Physically order numbers represented by place value blocks.

Stage II

A. Circle equivalent sets of 10 dots, tallies, x's, . . . and record number of sets of tens and number of ones left over.

B. Represent sets of tens and ones using place value block stamps or representational drawings, e.g., / / / . . . . . represents 35.

C. Illustrate rounding exercises through the representation of quantities on a place value pocket chart or abacus.

Stage III

A. Tell the place value of any digits in a number ≤ 1 million.

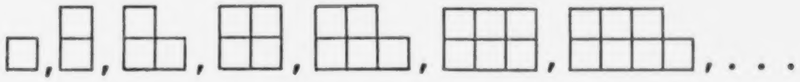
B. Round numbers to a given place value.

C. Order numerals.

o NUMBER THEORY

Stage I

A. Using cubes, represent even and odd numbers.



B. Using cubes, represent triangular and square numbers.

C. Using cubes, make all the possible rectangular arrangements to show factors of given numbers. Separate these into those that have just two possible rectangular arrangements and those that have more.

Stage II

A. Use dot arrays to represent all the factors of given numbers and classify them as prime or composite numbers.

B. Make factor trees for given numbers.

Stage III

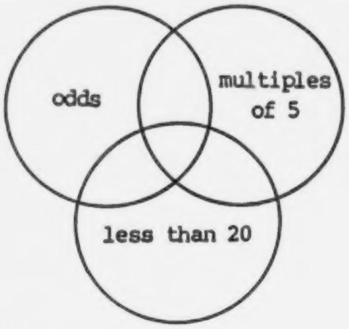
A. Complete number patterns, e.g., 2, 5, 8, __, __, . . .

B. List factors, multiples, G.C.F. (greatest common factor) and L.C.M. (least common multiple) of two or three numbers.

C. Discuss zero as even or odd, prime or composite, and multiple or factor.

D. Use divisibility rules for 2, 3, 4, 5, 6, 8, 9, and 10.

E. Using Venn diagrams, correctly place given numbers, e.g.,



F. Express and interpret powers of numbers using exponents.

G. Convert between standard and exponential notation.

OVERRIDING THEMES

Problem Solving. Counting is a very appropriate strategy for problem solving in the primary level. Many classic problems, like the "hand shake" problem, can be begun with the counting strategy.

Estimation and Reasonableness of Results. One area of estimation is to estimate quantity such as the number of people attending an event or the number of jelly beans in a jar. An appropriate technique for estimating quantities is sampling. Another is to repeatedly divide by two until an easily handled amount is reached.

Mental Computation. Counting on, counting back, and skip counting from any given starting number are appropriate oral activities for mental computation. An oral question like, "Which number has 4 tens, 3 ones, and 6 hundreds?" is appropriate for numeration. "Name the factors (or multiples) of 12," is an example of a number theory stimulus for mental computation.

Calculator/Computer Use. Calculators with constant functions are useful for skip counting. Entering numbers expressed in expanded notation such as 40 + 3 + 600 can help students establish the concept of place value.

INSTRUCTIOINAL MATERIALS

Abaci

Assorted Objects for Classification — different sizes, shapes, colors, textures, . . .

Attribute Blocks

Beans

Calculators

Chips

Computers

Counters

Geoboards

Place Value Block Stamps

Place Value Blocks

Place Value Mats

Place Value Pocket Chart

Templates for Number Writing and Pattern Blocks

Unifix Cubes

Strand: GEOMETRY

General Comments:

Geometry is a means of extending mathematics into the child's world. Children are familiar with items such as balls (spheres), wheels (circles), doorways and movie screens (rectangles) and can verbalize several properties about these items. When children begin to draw, their first drawings are open curves and then closed curves begin to appear. To an adult these drawings appear to be scribbles. As the children's perception of the world of geometry becomes refined, they begin to draw with lines and curves which are a two-dimensional representation of their three-dimensional world. At this time, provide children with the geometric models (representations such as circles, squares, prisms, and cubes) and allow them to orally discuss the similarities and differences about the models prior to the symbol (notation) or pictorial representation on the printed page. Geometry therefore, is a necessary and essential part of the mathematics program.

For learning to be effective, it must be more than a set of definitions. Students must be given the opportunity to talk about geometric properties and concepts. While the language of geometry is precise and provides an excellent opportunity to demonstrate to children its necessity, the mathematical language must become secondary to visual impact.

Strategies/Hints:

To develop an understanding of basic geometry concepts, students must be given an opportunity to do activities involving:

o **GEOMETRIC SHAPES AND TERMS**

Stage I

A. Feel geometric figures and shapes such as rectangles, squares, circles, triangles, cubes, spheres, and cones. Textured materials, such as sandpaper, enhance these experiences.

B. Sort, classify, and order geometric shapes.

C. Find examples of geometric figures in their world.

D. Develop the concept of curves by using string to define paths, by drawing paths between points in a variety of ways, and by feeling.

Stage II

A. Make associations between plane and solid figures by manipulating physical objects, e.g., squares and cubes, circles and cylinders, rectangles and cylinders, and circles and spheres.

B. Orally name drawings of simple geometric figures such as triangles, rectangles, squares, and circles. Figures may be physical models or pictures.

C. Draw representations of geometric figures.

Stage III

A. Classify triangles, quadrilaterals, rectangles, squares, and polyhedra according to stated criteria, e.g., number of sides, kind of angles, and number and shape of the faces (polyhedra).

B. Describe (orally and in writing) the shapes of physical models of polyhedra, polygons, circles, spheres using proper terminology, e.g., faces, vertices, edges, and diagonals.

o GEOMETRIC PROPERTIES

Stage I

A. Develop the concept of symmetry through activities such as completing the other half of a figure with mirrors and by folding and cutting paper figures.

B. Develop the concept of congruence by physically comparing several objects.

Stage II

A. Talk about special properties of simple geometric figures, e.g., compare/contrast rectangles and triangles, and discuss what is special about a circle.

B. Develop an understanding of congruence, similarity, and properties of geometric figures through translations (slides), rotations (turns), reflections (flips), and dilations (stretch and shrink). Models to use and talk about include paper folding, paper cutting, overlays, and scale drawings.

Stage III

A. Explain, orally and in writing, why two or more geometric figures are congruent or similar.

B. Use a protractor, compass, and straight-edge to construct a variety of geometric figures.

C. Draw three-dimensional objects on a two-dimensional surface.

o PERIMETER, AREA, VOLUME

Stage I

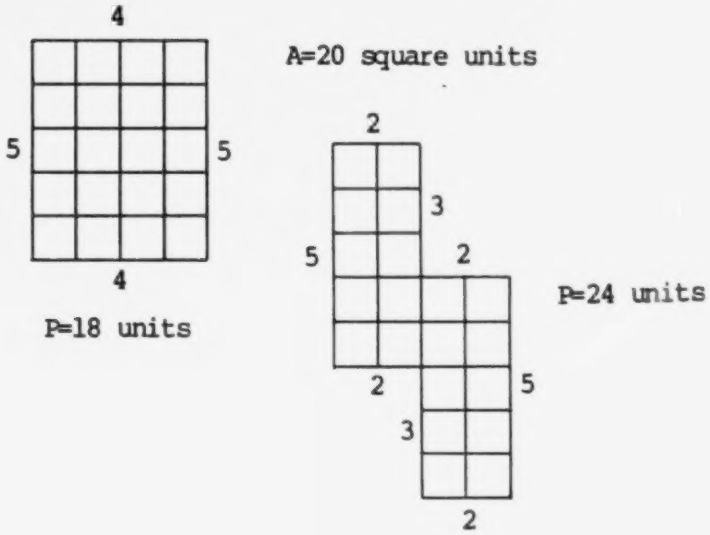
A. Develop the concept of perimeter by walking around the room or playground or by using string to go around shapes.

B. Develop the concept of area by covering or tiling a variety of flat shapes (surfaces) using congruent pieces for the tiling.

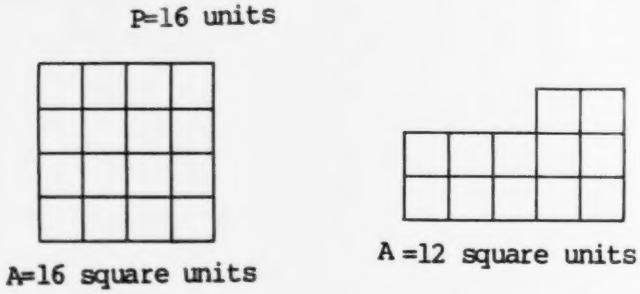
C. Compare volumes by pouring liquids from one container to another when the containers have different shapes and/or volumes.

Stage II

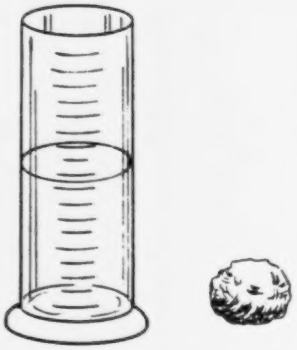
A. Predict the change or constancy of the perimeter of a given shape when the area stays the same but the shape is rearranged, e.g.,



B. Predict the change or constancy of the area of a given shape when the perimeter stays the same but the shape changes, e.g.,



C. Predict the amount of water that will be displaced by a given volume, e.g., How high will the water level rise if this rock is placed in the graduated cylinder?



Stage III

A. Apply formulas for perimeter, area, volume, and surface area to individual plane and space figures. It is not necessary or recommended that students memorize these formulas, nor can they use them with meaning until their conceptual development permits. This stage usually occurs during their junior high years.

OVERRIDING THEMES

Problem Solving. Introduction of each new concept should be made through a problem-solving situation, e.g., perimeter, area, volume.

Estimation and Reasonableness of Results. Students should estimate the distance around the room before determining a way to find the perimeter. Estimation is refined as students become more knowledgeable about available referents. Use of reference measures also makes possible a check on the reasonableness of results in area and volume.

Mental Computation. Much of the computation involved in geometry can be performed mentally, such as finding the perimeter and area of a rectangle or triangle and surface area and volume of a cube.

Calculator/Computer Use. Calculators should be used to handle computation with large numbers. Appropriate computer programs in the field of geometry should be selected for students to use.

INSTRUCTIONAL MATERIALS

Balls

Colored Water

Compasses

Congruent Two-Dimensional Pieces

Containers -- different shapes and volumes

Geoboards and Rubber Bands

Geometric Solids — spheres, cones, prisms, . . .

Globes

Graduated Cylinders

Graph Paper

Lego Blocks

Miras

Mirrors

Pattern Blocks

Protractors

Real World Setting — classroom, playground, furniture, playground equipment, . . .

Rocks

Scale Drawings

Straight Edge (rulers)

String

Tangrams

Tesselation (cut outs of shapes that tesselate)

Textured Materials -- sandpaper, fabric, . . .

Tinker Toys

Transparent Geometric Shapes (two dimensional)

Two- and Three-Dimensional Figures and Shapes

Strand: MEASUREMENT

General Comments:

This strand, often slighted in the K-8 mathematics curriculum, can provide many applications for other strands. The teacher must take an active role in teaching measurement through problems and activities that are meaningful to the student. Far too much is being taught abstractly — instead of measuring, students read about measurement. Students learn measurement best by doing appropriate practical measurement activities. Meaning for students is established through the use of manipulatives. Topics that should be included in this strand are money, time, temperature, linear measurement, area, mass, and volume. Do not have students convert from metric to customary measures or vice versa.

Strategies/Hints

To develop an understanding of measurement using both the metric and customary systems of measurement, students must be given the opportunity to do activities involving:

- **CUSTOMARY AND METRIC: LINEAR, AREA, VOLUME, AND MASS (WEIGHT) MEASUREMENT; TEMPERATURE**

  Stage I

  A. Compare and contrast two or more objects according to length, size, weight, or temperature, using terms such as longer-shorter, heavier-lighter, bigger-smaller, and hotter-colder.

  B. Use objects to display or "map" an area that has meaning to the students (such as a building, classroom, playground, and neighborhood), then draw a picture of the display. This develops position and distance relationships.

  C. Identify and use non-standard units of measure which have meaning to the students. Some examples are hand span, length of foot or arm, height, length of pencil, mass (weight) of a book, and height or width of their desk. This establishes the need for standard units of measure.

  Stage II

  A. Measure "common" objects with teacher-selected measuring devices. This should gradually lead to the student selecting the most appropriate measuring device and unit of measure. This should be done with both customary and metric units of measure.

B. Discover equivalent relationships between units of measure, (e.g., 3 feet = 1 yard, 16 tablespoons = 1 cup, 3 teaspoons = 1 tablespoon, 1 cubic centimeter = 1 millimeter, and 10 centimeters = 1 decimeter).

C. Estimation skills should be used to develop an understanding of conversions within each of the two systems and the relationship that exists between them, (e.g., one meter is about a yard, a quart is about a liter, and one kilogram is about two pounds).

D. Interpret and determine distances using a scale given on a map.

E. Recognize "critical" points on a thermometer, e.g., freezing, boiling, and comfort zone.

Stage III

A. Make a decision on how precise a measurement should be and make the measurement to that precision.

B. Recognize that any stated measure has some inaccuracy, and that the amount of inaccuracy depends on how the measurement was made.

C. Select the "best" unit of measure and "best" measuring device.

D. Choose an appropriate unit of measure and appropriate formula to find area of simple polygons and surface area of polyhedra.

E. Find the perimeter, area, and, if applicable, surface area and volume of irregular geometric figures.

F. Read a thermometer in both Fahrenheit and Celsius.

o MONEY

Stage I

A. Use actual coins or replicas to recognize, name, and compare the coins.

B. Choose coins necessary for purchase of a given item, recognizing that there may be many possible choices for the same purchase.

Stage II

A. Recognize decimal form for coin value, e.g., 25¢ = $.25

B. Use money or replicas to make change or to make a purchase.

Stage III

A. Compute with amounts of money.

B. Recognize if accurate change has been given for a purchase.

o TIME

Stage I

A. Develop the concept of time using familiar terms such as before-after, first-next, first-last, daytime-nighttime, morning-afternoon, and sooner-later.

B. Read digital and conventional (analog) clocks and calendars.

Stage II

A. Develop the concept of a.m. and p.m. by letting these terms replace "morning," "afternoon," "evening," nighttime," . . .

B. Sequence events through a variety of activities including storytelling, time line, and picture arrangement, e.g., seasons, growth of a seed to plant. Units of time can be extended to day and year.

Stage III

A. Identify different time zones on maps and globes.

B. Discuss different time zones with respect to phone calls between time zones and actual flight time compared to departure and arrival times.

OVERRIDING THEMES

Problem Solving. Measurement activities should not be separated from problem solving. Measurement problems need to be acted out or simulated.

Estimation and Reasonableness of Results. Students need to estimate before doing any measurement. This not only develops an intuitive frame of reference for measures, but also engages the interest for the actual measurement activity. Students can estimate length, weight, area, volume, and temperature (with both metric and standard units) at the end of the measurement activity. The estimate can serve as a check on the reasonableness of the outcome. Students can also estimate amounts of money needed for a purchase. They need to recognize when it is advantageous to overestimate and underestimate amounts of money.

Mental Computation. Students need oral work with converting from one unit to another within the same system. This can be incorporated as an opening activity for periodic review in conjunction with any mathematics lesson.

Calculator/Computer Use. Computer programs must permit a range of answers in an estimation situation. Logo provides an excellent opportunity for estimating size of angles and line segments.

INSTRUCTIONAL MATERIALS

Assorted Objects to Compare and Contrast

Assorted Physical Objects to Use as Nonstandard Measuring Devices

Blocks

Boxes — made or to make

Calendars

Clocks --analog and digital

Colored Pencils

Colored Water

Containers -- various sizes and shapes

Cubes -- centimeter and inch

Geoboards

Globes and Maps

Graph Paper — square centimeter and square inch

Linear Measuring Devices — metric and customary

Maps which identify Miles per Inch, Kilometers per Centimeter

Measuring Spoons and Cups

Mileage Charts

Money or Replicas

Pentaminoes

Polygons — regular and irregular

Polyhedra

Rice

Sand

Scales and Balances

Sundials

Thermometers — Celsius and Fahrenheit

Time Tables for Travel — air, rail, bus

Transparent Grids — square centimeter and square inch

Unmarked Measuring Devices

Variety of Shapes — regular and irregular polygons, circles

Volumetric Measuring Devices — metric and customary

Weights/Masses — customary and metric

Strand: PROBABILITY AND STATISTICS

General Comments:

In developing probability and statistics concepts for elementary students, the major emphasis should be on collecting, organizing, and displaying data. The students must be given opportunities to do all three through a series of meaningful activities. One of the goals of this curriculum model is to emphasize a participatory role for students while they learn. The activities in this strand provide many opportunities for this type of learning. The activities also allow students to apply previously learned skills such as numeration, computation, and geometry.

Strategies/Hints:

To develop an understanding of basic concepts of probability and statistics, students must be given an opportunity to do activities involving:

o DATA COLLECTION, ORGANIZATION, AND INTERPRETATION

Stage I

A. Obtain data about student personal characteristics (attributes).

B. Record the data obtained. The teacher should record the data while discussing with the students how it should be organized.

Stage II

A. Obtain data from sources immediately available to the students such as birth months, height, and favorite color.

B. Determine how the data should be organized. The students should make this decision and then make inferences from the data.

Stage III

A. Determine the type and source of data that is necessary to solve a problem. Data are collected and organized by students in a manner that will convey greatest understanding to the reader. Concluding statements based on the data are written by students.

B. Read and interpret tables, charts, graphs, and advertisements that contain incomplete or distorted data.

C. Draw and evaluate conclusions from tables, charts, graphs, and advertisements.

D. Evaluate others' conclusions from tables, charts, graphs, and advertisements.

o **SIMPLE PROBABILITY**

Stage I

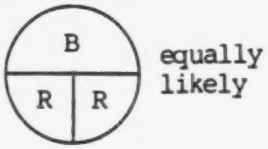
A. Use students' environment to make simple predictions such as - "Will we have school tomorrow?", "Will it rain?", and "How many students are eating lunch today?".

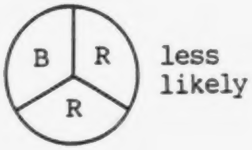
B. Discuss the validity of the predictions.

Stage II

A. Conduct experiments using items such as spinners, coins, marbles, and dice. Record the outcomes from numerous experiments. Describe the outcomes that occur as a fraction of all the possible outcomes and compare the likelihood of two events.

B. Recognize that a spinner landing on blue can be a less likely or an equally likely outcome if there are two red sections for every blue section, e.g.,





C. Discuss the difference between the words "or" "and." When rolling a die an even or a prime number is more likely than rolling an even and a prime number.

D. Explore and record results of drawing with and without replacement. For example, the likelihood of drawing two red marbles from a bag of colored marbles depends on whether or not marbles are replaced after each selection.

Stage III

A. Use a variety of data sources to continue the development of previously learned predictive skills and to determine the probability of an event.

o COUNTING STRATEGIES

Stage I

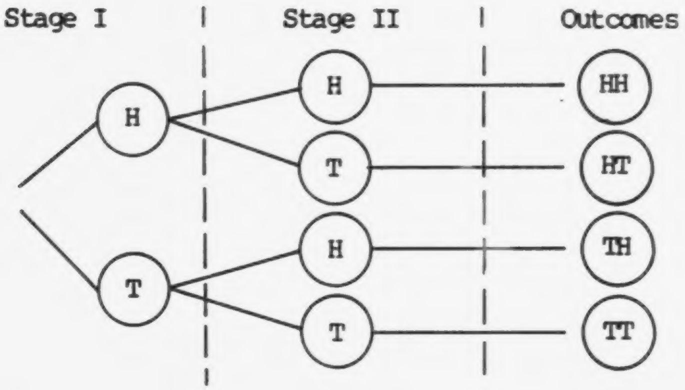
A. Identify different ways to arrange discrete models such as books, students, and letters of the alphabet.

B. Record different possible arrangements using discrete models, e.g., three students.

Stage II

A. Use tables to organize outcomes from data gathered through surveys, e.g., favorite television show color, food, and book.

B. Construct tree diagrams to represent the total number of outcomes (the sample space) for a two-stage experiment.



Stage III

A. Use sampling to determine outcomes when counting is not feasible.

B. Use tree diagrams to identify possible outcomes (sample space) for multi-stage experiments.

C. Determine the number of different arrangements when the order of the objects is important and when the order is not important.

o MEASURES OF CENTRAL TENDENCY

Stage I

A. Collect numerical data from student attributes and interest surveys. Data could be ordered from greatest to least with tally marks used to indicate duplication of values. Examples of data sources to be included are height, hand span, number of hours of television watched, age (in months), or heartbeat.

B. Have students discuss answers to a variety of questions such as what is the greatest, least, most common, and middle value as well as the difference between the greatest and least value.

Stage II

A. Arrange data in a stem and leaf plot, e.g., 36, 23, 35, 29, 37, 37

Tens	Ones
2	3 9
3	5 6 7 7 .

In addition to examples of data sources previously mentioned, include sports statistics, temperature readings, and rainfall measures.

Stage III

A. Calculate range, mean, media, and mode from ungrouped data.

B. Organize data into a frequency distribution. Students use calculators to compute range, mode, and mean from the frequency distribution.

o GRAPHING

Stage I

A. Move objects or markers on a number line or scale (thermometer) from one location to another based on oral discussions and directions.

B. Move objects or markers on a grid from one location to another by following oral directions, e.g., move three units to the right, and move three units up, two units to the left and four units up, . . .

Stage II

A. Name a location by means of an ordered pair.

B. Locate ordered pairs on graphs or grids in a game situation such as "Hurkle" and "Battleship."

Stage III

A. Locate, name, and plot ordered pairs using "x" and "y" coordinates.

B. Solve and graph simple linear equations using □ and △ (in later grades "x" and "y") as representations for variables.

OVERRIDING THEMES

Problem Solving. Gather and organize data collected from surveys. Use information of interest to students. Sport statistics, temperature, rainfall, hours of television watched, heartbeat, and age (in months) are examples of good sources of problems.

Estimation and Reasonableness of Results. Use the students' environment to make simple predictions such as "Will we win our softball game tomorrow?", "How many students will participate in play-day activities?", and "How many students in our class are born in the month of January?". To check for the reasonableness of results, the mean of a set of data should not be greater than the largest number or smaller than the smallest number.

Mental Computation. Determine the mean and range of a small set of data. Determine the intervals to be used in preparation for graphing a set of data.

Calculator/Computer Use. Use calculators to compute range, mode, median, and mean of a large set of data. Use computers to organize frequency distribution of large sets of data.

INSTRUCTIONAL MATERIALS

Advertisements

Beans

Chips -- colored

Coins

Data or Information Organized into Tables, Charts, and Graphs

Dice

Discrete Models -- books, balls, . . .

Graph Paper

Marbles — colored

Markers — counters, beans, chips, . . .

Measuring Devices — linear, volume, mass/weight, thermometers

Number Lines

People

Spinners

Stop Watches or Clocks

Transparent Grids

Unifix Cubes

Strand: RATIO, PROPORTION, AND PERCENT

General Comments:

Ratio, proportion, and percent are often neglected at the junior high level when an inordinate amount of time is spent with whole number computation. The ability to apply ratio, proportion, and percent is an important problem-solving skill. It is essential that students have an opportunity to internalize fraction concepts before they see the larger concept of ratio. Therefore, ratio, proportion, and percent should not be taught at too early an age, nor should they be taught in abstraction.

In a fraction, the "part" is compared to the "whole." In a ratio, the "part" is not only compared to the "whole," but also to the other "part." The latter interpretation of ratios is confusing to students when comparing adding fractions to adding ratios.

Labeling ("part" and "whole" and "units of measurement") is crucial to meaningful applications of proportions and percents. Algebraic interpretation of the three cases of percent problems is better handled by setting up and solving proportions rather than an algebraic equation, e.g., 15 = 25% of what number? $15 = 0.25 \times n$ versus

$$\frac{15}{n} = \frac{25}{100}.$$

Another confusion for students is the use of the term "percentage." We recommend the word "amount" be substituted for "percentage," e.g., rate x base = amount.

Strategies/Hints:

To develop an understanding of ratio, proportion, and percent, students must be given an opportunity to do activities involving:

o RATIO

Stage 1

A. Model situations such as, "There are three red chips for every four blue chips. If I have six red chips, how many blue chips do I need?"

B. Experience ratios in a "discovery" mode. Through the use of manipulatives students could determine the answer to "How many brown candies are there for each green one?", "How many orange candies are there for each yellow one?", . . .

C. Picture ratios using graph paper, stick figure drawings, money, . . .

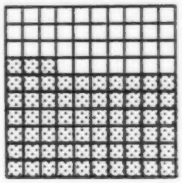
Stage II

A. State ratios with which they are familiar. "I need three cans of water for each can of orange juice concentrate" (miles per gallon, miles per hour, . . .).

B. Use a table format for recording equivalent ratios.

cans of water	3	6
cans of orange juice concentrate	1	?

C. Model percents as ratios using 10 x 10 grids.



63% or 63:100

Stage III

A. Work with the written form "A to B" or "A:B" but are not given the fraction-like form "$\frac{A}{B}$".

B. Extend the "A to B" for a ratio to "A to B = C to D" or "A:B::C:D" for equivalent ratios.

C. Write percents in ratio form.

D. Write equivalent percents as ratios in table form using them to solve problems.

o PROPORTION

Stage I

A. Model equivalent ratios and use the table form to record the outcomes. This is a transition from ratio to proportion.

B. Use graph paper to show equivalent ratios.

C. State equivalent ratios from given models or pictures.

Stage II

A. Given equivalent ratios, model the proportion using graph paper, stick figures, tables, drawings, . . .

B. Model scale drawings and map distances as proportions. A motivational application of scale drawings is to superimpose a transparency of a grid over a graphic and then reproduce the graphic using a grid where each cell is enlarged.

C. Write equivalent ratios using the form "A to B = C to D" or "A:B::C:D."

D. Write equivalent ratios in the fraction-like form $\frac{A}{B} = \frac{C}{D}$ which is developed as an extension of the table format. (Emphasize the labeling of numbers as "part," "whole," or appropriate "units of measure" to keep relationships clear.)

Stage III (Note: This content and appropriate activities should be delayed until the junior high level.)

A. Introduce "cross products" as a way of simplifying proportions.

B. Use proportions to simplify percent problems.

C. Use proportions to simplify problem situations, e.g., "You have 16 girls and 24 boys to put in two groups. If the ratio of girls to boys remains the same, how many girls and how many boys will be in each group?"

o PERCENT

Stage I

A. Shade squares in 10 x 10 grids to show any number per hundred (oral name to model). Also, given a partially shaded grid, tell how many per hundred are shaded (model to oral name).

B. Use 10 x 10 grids as manipulatives by folding the grid in half or thirds or fourths and counting the number of units in one half, one third, two thirds, one fourth, three fourths. (In the case of the thirds, three partial units need to be put together to make a unit and show that $\frac{1}{3} = 33\frac{1}{3}$ out of 100.)

C. Use place value blocks or interlocking cubes as manipulatives for percent. The hundred flat is renamed as the unit, the ten long as the tenth, and the ones unit as the hundredth.

Stage II

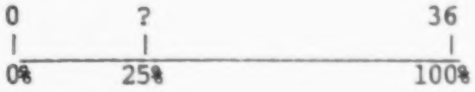
A. The % symbol should not be introduced until students are comfortable going from the model to the verbalization (oral) of amount per hundred.

B. Write the percent for the shaded 10 x 10 grid. Students estimate the percent of a shaded bar or circle or other objects.

C. Using number scales, students judge where appropriate quantities or percents would be located. There are three cases of percent to consider:

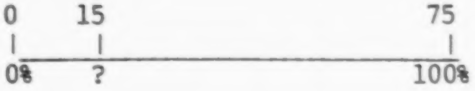
Case I

What is 25% of 36? If on a numberline, 36 is equivalent to 100%, what number would be equivalent to 25%?



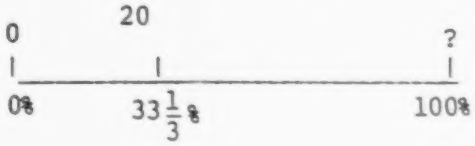
Case II

Fifteen is what percent of 75? If on a number line 75 is equivalent to 100%, what percent would be equivalent to 15?

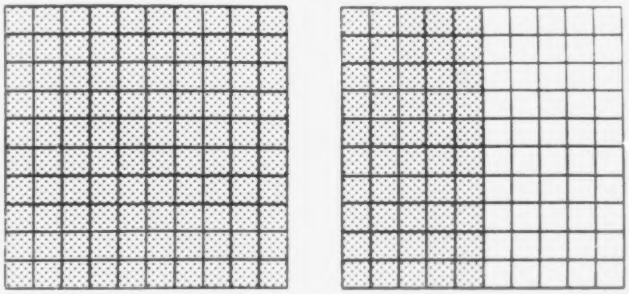


Case III

Twenty is $33\frac{1}{3}$% of what? If on a number line, 20 is equivalent to $33\frac{1}{3}$%, what number would be equivalent to 100%?



D. Experience matching graphics to numerals and numerals to graphics when percents are greater than 100 and less than one, e.g., 150%.



Stage III

A. Use proportions for problem-solving situations with the following activities:

1. change fractions to equivalent decimals and vice versa.

2. change decimals to equivalent percents and vice versa.

3. change fractions to equivalent percents and vice versa.

B. Solve the three cases of percent problems.

C. Apply proportions to interest problems, percent of increase or decrease, and discount. Newspaper and magazine articles should be constantly used at this level as sources for applications in sports, banking, stocks, science, . . . The calculator is a necessity for this level.

OVERRIDING THEMES

Problem Solving. These concepts have direct relation to everyday applications of consumer and personal mathematics including banking, purchasing, taxes, and sports. It is unrealistic to consider teaching these topics outside the problem-solving framework. Problem-solving strategies such as "acting out" and "drawing a diagram" are especially useful for these concepts.

Estimation and Reasonableness of Results. Tips, best buy, sales tax, and discounts are all practical applications for estimation. The development of the concept of percent can be facilitated when students estimate the percent of a region that is shaded. Students should particularly be encouraged to check the reasonableness of results for the three cases of percent problems because multiplying when one should divide frequently leads to an unreasonable answer.

Mental Computation. Being able to mentally compute with proportions and percents is a valuable skill for adults. Calculating sales prices given the percent of discount or best buy are examples of common activities. Mental computation with percents is facilitated when students memorize common fraction-percent equivalences and then relate any percent to these "nice percents." Decimal-percent equivalences are naturals for mental computation.

Calculator/Computer Use. The computational power of the calculator eliminates the drudgery associated with proportion and percent problems and enables the student to focus on the "heart" of the problem. This will also be the focus when mathematics is used outside the school setting. Students need instruction in the use of the percent key.

INSTRUCTIONAL MATERIALS

Attribute Blocks

Calculators

Candy — colored or flavors

Chips — colored

Colored Blocks

Graph Paper

Grids (10 x 10)

Interlocking Cubes

Money

Newspapers and Magazine Articles

Paper Plates

Place Value Blocks

Scale Drawings

Small Toys

Strand: WHOLE NUMBER COMPUTATION

General Comments:

The purpose of the following Stage I and II activities is to build for understanding that will later be applied when students use the textbooks. These activities are time consuming and should not be rushed. Meaningful instruction of concepts, place value, and computational algorithms requires a three-dimensional model such as place value blocks. Time and instruction are necessary for teachers and students to become proficient in manipulating and building the algorithms. Time spent manipulating — at the concrete level — to develop the concept will enhance the student's learning and this will require less time teaching the algorithm at the abstract level. This also avoids endless practice with computation. Having students manipulate the model individually and in small groups is more effective than when the teacher demonstrates the manipulation of the models.

Problem solving with whole numbers or decimals should not be delayed until students reach the advanced level. Since problem solving is an integral part of instruction to motivate the learner and make mathematics relevant, computation should not be taught in isolation.

Strategies/Hints:

To develop an understanding of the basic concepts of whole number computation, students must be given the opportunity to do activities involving:

o BASIC FACTS

Stage I

A. Build models for the addition and multiplication facts using concrete objects and demonstrate understanding of operational concepts by talking about them. For example — given five beans, the students are asked to explore and discover how many combinations, arrangements, or groupings can be formed for five. This is an intuitive stage when the student recognizes, internalizes, and can state that five is three and two or four and one. The objects are not used simply to illustrate the abstract symbols, 3 + 2 = 5. The abstract symbols should be delayed to allow students opportunities to internalize the concepts.

B. Use counters effectively, e.g., when counting on. If a student uses blocks as counters to solve the problem, "Sue has five stickers and received three more, how many did she have?", The student should respond, "Five blocks, six, seven, eight," rather than "One, two, three, . . . , eight."

C. Spending time on subtraction facts and division facts is counterproductive. The relationship of subtraction to addition and of division to multiplication should be taught simultaneously.

Stage II

A. Interpret representational drawings of facts, and draw the facts.

B. Provide the students with directions and instruction on proper interpretation and use of dots, tallies, and objects that represent the facts as textbooks traditionally have students interpret such drawings of the facts.

Stage III

A. Use number lines to demonstrate understanding of the basic facts.

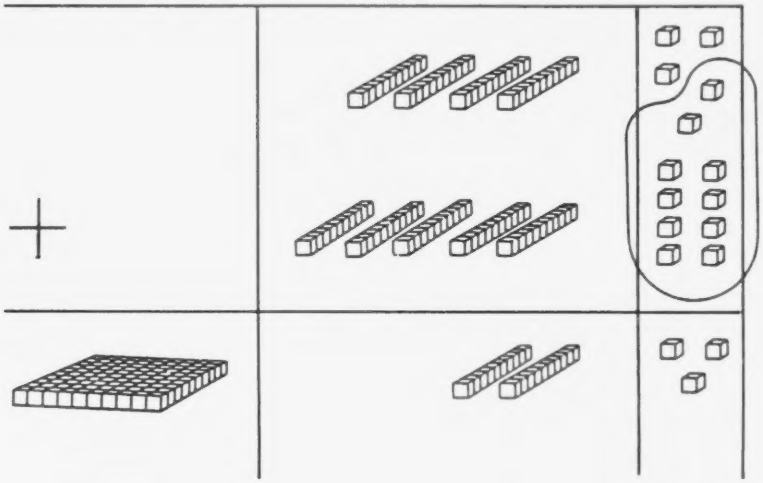
B. Quickly and effectively complete written and oral basic fact statements and apply them to problem-solving situations.

o ALGORITHMS

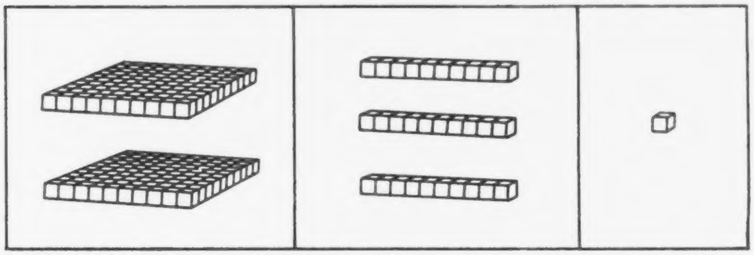
Stage I

A. Build models using place value materials and demonstrate understanding of the algorithm through verbalization. Students need instruction on how to use place value blocks. Students must be given time for exploration to internalize the meaning/ properties of the concept. Problem solving should be used to introduce each algorithm. Examples are:

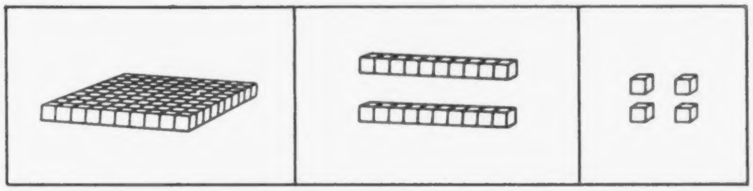
addition:*



subtraction:* (One strategy. Others exist)

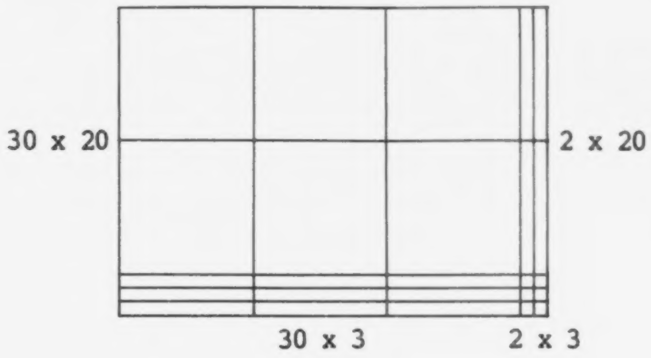


"Take away one hundred seven..."



multiplication:* (One strategy. Others exist)

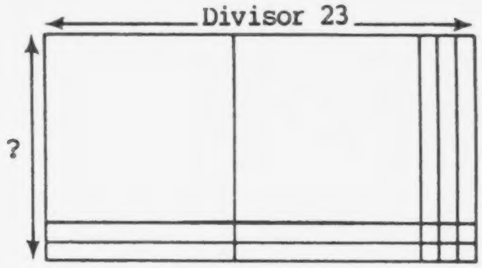
Build 23 x 32 and separate it into four regions to represent the four partial products.


30 x 20
2 x 20
30 x 3
2 x 3

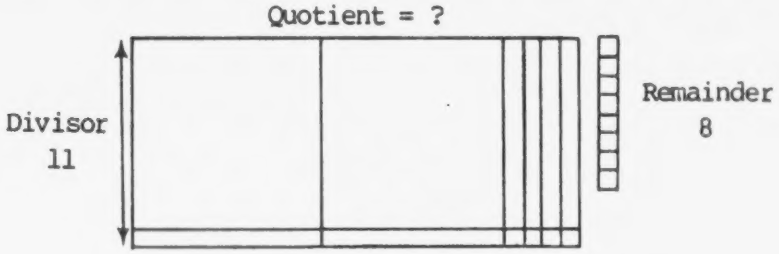
$$\begin{array}{r}23\\ \underline{x32}\\ 6\\ 40\\ 90\\ \underline{600}\\ 736\end{array}$$

division:* (One strategy. Others exist)

ex. 1)
What is 276 divided by 23?



ex. 2)
What is 272 ÷ 11?


Quotient = ?
Divisor
11
Remainder
8

* adapted from Base Ten Mathematics,
Interludes for Every Math Text,
by Mary Laycock,
Activity Resources Co., Inc.
Hayward, CA., 1977.

Stage II

A. Interpret and draw/create representations of algorithms. This step is critical to the total development of the algorithm and should not be rushed. Bundling straws, toothpicks, tongue depressors, Unifix cubes, Multilink cubes, and graph paper are models that can be used instead of, or in addition to, place value blocks.

Stage III

A. Use the algorithms in the abstract and apply to problem-solving situations.

B. Demonstrate understanding of the algorithm by writing a description of the process.

C. Students should not perform addition and subtraction on numbers larger than five digits. Factors should not exceed three digits. Divisors should be limited to two digits and dividends to four digits. (Note: Any computation beyond these limitations should be done on the calculator.)

OVERRIDING THEMES

Problem Solving. Problem solving should begin in the primary grades, be illustrated with place value blocks, and be focused around strategies to choose the correct operation. Problem solving should not be limited to the "key word" strategy.

Estimation and Reasonableness of Results. Estimation should be taught independently and should not be limited to rounding. Estimation and reasonableness of results should be constants of instruction with estimating the answer to open an activity and checking reasonableness to close it.

Mental Computation. Mental computation should be a routine part of most class periods. Strategies such as "making a ten" need direct instruction, e.g., $35 + 8 = (35 + 5) + 3$ and should be illustrated with models. Instruction can begin in the primary grades with the basic facts and build to chained exercises with several operations.

Calculator/Computation Use. Calculators and computers should be used for problem solving and "number crunching" with whole numbers. (Computers have definite advantages for drill and practice but computer use needs to go beyond this to problem solving.) Computer programs which assist in whole number operation instruction must use representational models.

INSTRUCTIONAL MATERIALS

Abaci

Beans

Calculators

Chips

Computers

Counters

Graph Paper

Multilink Cubes

Place Value Blocks

Place Value Boards

Popsicle Sticks

Straws

Tongue Depressors

Toothpicks

Unifix Cubes

Strand: DECIMAL FRACTIONS

General Comments:

The purpose of the following Stage I and II activities is to build a foundation for understanding that will later be applied when the students use the textbook. These activities are time consuming and should not be rushed. Meaningful instruction of decimal concepts, place value, and computational algorithms requires a three-dimensional model such as place value blocks. Time and instruction are necessary for the teacher and the students to become proficient in manipulating and building the algorithms. It is important that students develop a strong conceptual base before the algorithms/procedures of computation are taught. This conceptual base can be best learned through manipulation of models by students. This also requires a reallocation of time. When more time is spent developing a strong conceptual base during the early years, less time will be required teaching the algorithm/procedures in subsequent grades.

Problem solving with decimals should not be delayed until students reach Stage III. Since problem solving is an integral part of instruction to motivate the learner and make mathematics relevant, computation of decimals should not be taught in isolation.

Strategies/Hints:

The following experiences are best learned when introduced at the concrete level through problem-solving activities. Paper-pencil activities are minimized at this stage. To develop an understanding of basic concepts of decimals, students must be given an opportunity to do activities involving:

o **CONCEPT AND PLACE VALUE**

Stage I

A. Manipulate models, such as place value blocks, to begin understanding size relationships and place value. Considerable time should be spent placing the model pieces in the correct places and saying the names of the pieces, both as individual digit names and as the entire number.

e.g., "4.621 = 4 ones
6 tenths
2 hundredths
1 thousandth

4.621 is four and six hundred twenty-one thousandths."

B. Build a model after given the decimal fraction name and the mixed fraction name. Additionally, students should relate decimal names to the real world by finding examples. For example, given dimes and pennies, students should recognize their value and be able to relate them to tenths and hundredths.

C. Create their own decimal models of tenths and hundredths from paper, straws, Unifix cubes, or Multilink cubes. In addition, students should make oral comparisons of sizes of pieces. Build models of decimal fractions and orally state the names of the modeled numbers.

D. After place value has been understood, all rounding and comparative activities can successfully be introduced by using manipulatives.

E. Recognize the patterns occurring in the place value names on each side of the unit's place (tens, tenths, hundreds, hundredths, . . .).

Stage II

A. Interpret and create representational drawings of decimals and compare them using graph paper. Students should be able to perform chip-trading activities that have been related to decimal place values, e.g., poker chips, the abacus with **different colored discs** for each place value, different colored beads, . . .

B. Trade identical objects that are distinguished only by the placement of the object. Instructional materials include place value charts and abaci which have discs of the **same color** for all place values.

C. Build the models and name them through written and oral language during all activities. Comparing and rounding activities must continue to occur.

Stage III

A. Write the name for decimal fractions and mixed decimal fractions given in digit and oral form. Student should be able to orally read and record the digits of a number given in written form. The relationships among decimal names should be established easily and fluently without the use of models or physical representations.

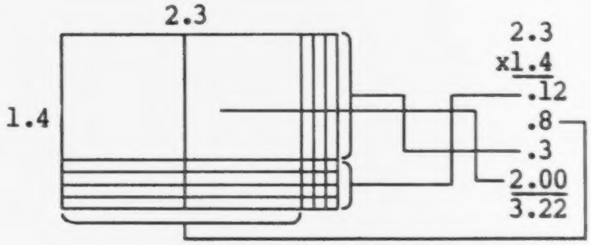
o OPERATIONS

Stage I

A. Add and subtract decimal fractions and mixed decimal fractions using models. Exploratory time must be available for students to apply their understanding of trading and regrouping.

B. Use models to multiply decimals. Place value blocks can be successfully used for factors of mixed fractions to tenths, where the Flat is used to represent ones, the Rod is tenths, and the Unit Cube is hundredths.

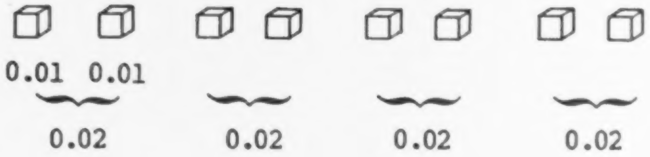
example: 2.3 x 1.4 = 3.22*



(Note: The teacher should prepare workable problems in advance.)

C. Model division using place value blocks. Divisors should be limited to whole numbers and dividends to hundredths so that the idea of placing the decimal fraction (dividend) in equal groups (divisor) can be demonstrated.

example: $4\overline{)\,.08}$



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*Base Ten Mathematics, Mary Laycock

Some students might see the relationship of "quotient times divisor equals dividend." This insight should not be forced at this level, but left to the students who are ready.

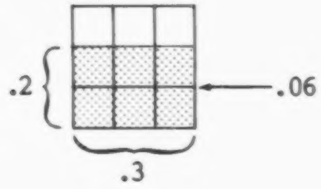
Stage II

A. Interpret and draw/create representations of the algorithms. This step is critical to the total development of the algorithms.

B. Draw representations of the place value blocks and attach the symbol to show the algorithms for addition and subtraction.

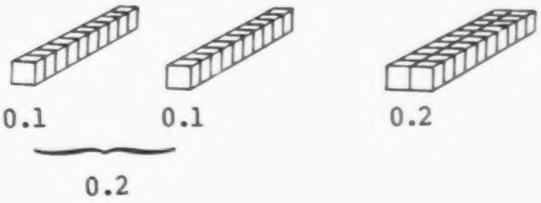
C. Use grids (graph paper) on a transparency to show the products for multiplication.

example: 0.2 x 0.3



D. Draw place value blocks, organizing them to show equal groupings to represent divison problems.

example: $2\overline{)\,0.4}$



Stage III

A. Use the algorithms in the abstract and apply them to problem-solving situations.

B. Demonstrate understanding of the algorithms by writing a description of the process and by using the rules of decimals.

C. Computations done by paper and pencil should not exceed thousandths. Any computations beyond this limitation should be done on the calculator.

OVERRIDING THEMES

Problem Solving. Problems dealing with money need to deal with converting from "¢" to "$" notation. Banking and metric measurement are other sources of problem solving with decimals.

Estimation and Reasonableness of Results. When estimating with decimals, rounding or truncating to the nearest whole number is essential for adding and subtracting. Estimating by classifying decimals as close to zero, one-half, or one needs instruction. Placement of the decimal point is the key to estimation when multiplying and dividing. Students should estimate the decimal fraction for the shaded portion of a square.

MENTAL COMPUTATION. Mental computation with money should be taught in conjunction with whole numbers.

CALCULATOR/COMPUTER USE. Once students have internalized placement of the decimal point, the calculator should be used routinely for all operations. Students need instruction in periodic decimals and truncation when dealing with money on the calculator.

INSTRUCTIONAL MATERIALS

Abaci With Different Colored Discs

Abaci With Discs of the Same Color

Base Ten Stamps

Beads — colored

Calculators

Chips

Computers

Graph Paper

Grids (10 x 10)

Metric Measurement Devices

Money

Number Lines

Place Value Blocks

Place Value Charts

Sale Advertisements

Strand: FRACTIONS

General Comments:

Fractions dominate the mathematics curriculum in upper elementary/ middle school. In spite of the large amount of teaching time devoted to fractions, student deficiencies in understanding this concept are prominent. Rote procedures are applied to innappropriate situations. The Second NAEP Mathematics Assessment included the following exercise:

"Estimate the answer to $\frac{12}{13} + \frac{7}{8}$. You will not have time to solve the problem using paper and pencil."

The subsequent response pattern indicates little understanding of the concepts of fraction per se and the addition of fractions. Only 24 percent of the 13 year olds were able to answer this question correctly.

Answers	Percent Responding (Age 13)
1	7
2	24
19	28
21	27
I don't know	14

It is logical to assume that the remaining 76 percent of the students would have benefitted from concept development through active student involvement with various types of manipulatives, through extensive experiences drawing fractional parts and interpreting fractional drawings, and through the encouragement of appropriate oral language.

Research indicates that children who can represent a concept with models and oral language and very fluently translate from one type of representation to the other are better prepared to learn to use written symbols as a means of representing the concept.

Fraction instruction has been typically dominated by a short lesson on the concept of fraction followed by lengthy lessons with fraction operations (algorithmic manipulation). Students' lack of success with these operations at grade 4 leads to extensive practice at grade 5, more of the same at grades 6 and 7, and a remedial approach at grade 8 and beyond. Because of these problems, current research, technological advances, and other reasons, it is strongly recommended that formal instruction on operations with fractions having unlike denominators should be delayed until sixth grade. Even then, denominators should be limited to those commonly encountered in situations such as measurement activities.

Strategies/Hints:

To develop an understanding of fractions, students must be given an opportunity to do activities involving:

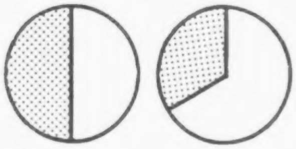
o CONCEPT, EQUIVALENCE, ORDERING

Stage I

A. Divide real objects (food, paper) into halves, thirds, fourths, . . .

B. Use fraction circles to match models to numerals and numerals to models before learning the written names of fractions.

C. Compare equivalent and non-equivalent fractions of real objects, rods, or fraction circles, e.g.,



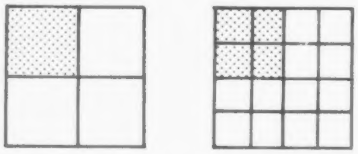
$\frac{1}{2} > \frac{1}{3}$

Stage II

A. Match picture to oral name, oral name to picture, picture to numeral, and numeral to picture using pictorial representations of fractions (of regions and sets).

B. Compare and order pictorial representations of both equivalent and non-equivalent fractions.

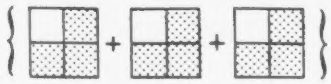
C. Recognize and draw pictorial representations of both equivalent and non-equivalent fractions, e.g.,



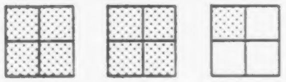
1/4 = 4/16

D. Compare and order representations to show the equivalence of improper fractons to mixed numerals.

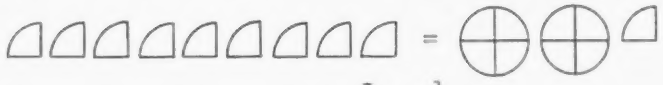
E. Recognize and draw representations to show the equivalence of improper fractions to mixed numerals.



$= \frac{9}{4} = 2\frac{1}{4}$



$= 2\frac{1}{4}$



$\frac{9}{4} = 2\frac{1}{4}$

F. Mark common, like fractions (same denominators) and mixed numerals on a number line.

Stage III

A. Read and write numerals for fractions.

B. Express in written form and show the comparison of fractions having like denominators.

C. Describe the meanings of numerator and denominator in practical situations.

D. Express fractions in higher and lower terms.

E. Compare fractions having unlike denominators.

o OPERATIONS

Stage I (Note: Not formally taught until 5th grade)

A. Use fraction circles or bars to discover the relationship and verify mathematical sentences such as:

$\frac{1}{2} + \frac{2}{4} = 1$

$\frac{1}{3} + \frac{1}{3} < 1$

$\frac{5}{6} + \frac{1}{4} > 1$

$3 \times \frac{1}{4} < 1$

Stage II

A. Interpret and represent the addition and subtraction of like fractions and mixed numerals using the number line or drawings. Attach the symbol for their sum or difference.

B. Interpret and represent the multiplication of fractions using rectangular regions.

Stage III

A. Compute with fractions and mixed numerals and express the answer in simplest terms.

B. Apply operations with fractions to the solution of measurement and probability word problems.

C. Describe (aloud and in writing) the process of any operation on fractions.

D. Express common fractions as decimals and vice versa.

OVERRIDING THEMES

Problem Solving. Problem situations should be illustrated by drawing models (fraction circles, bars, fraction of a set, number line). Probability is a good source for problem situations.

Estimation and Reasonableness of Results. Students should estimate the shaded fraction of a bar and a marked fraction on a number line. When estimating with fractions, classify each fraction as close to zero, close to one-half, or close to one. For fraction of a number, use the compatible number strategy (e.g., $\frac{1}{3}$ of 62 $\approx \frac{1}{3}$ of 60 or 20).

Mental Computation. Students can mentally compute most operations with simple fractions, especially addition and subtraction with like denominators and the fractional part of a number (e.g., $\frac{3}{4}$ of 12).

Calculator/Computer Use. For computation on the calculator, students need instruction in fraction decimal equivalence, periodic decimals, and the use of memory. CAI (Computer Assisted Instruction) programs for fractions should use representational models.

INSTRUCTIONAL MATERIALS.

Assorted Objects to Cut or Group -- beans, corn, containers, lids, sticks, pies, . . .

Computers

Counters

Cuisenaire Rods

Flannel Boards and Fraction Pieces

Fraction Bars

Fraction Pieces — circles and regular polygons

Number Lines

Paper for Folding

Place Value Blocks

Strand: INTEGERS

General Comments:

The purpose of the following activities is to develop an understanding of integers. This is the first time that students have been exposed to a number system which includes numbers less than zero. In most situations, students will not formally work with integers until middle school/junior high grades. However, the concept of integers may be introduced much earlier.

Strategies/Hints:

To develop an understanding of the basic concepts of integers, students must be given the opportunity to do activities involving:

- **CONCEPT OF INTEGER INCLUDING LANGUAGE AND NOTATION**

Stage I

A. Talk about situations where they have seen, heard, or been aware of numbers less than zero, e.g., thermometers, football yardage, sea level - above and below.

B. Observe a method for recording this information. Develop correct terminology, e.g., "positive" and "negative" instead of "plus" and "minus" in oral communication and recognize the written word.

Stage II

A. Order integers on a number line. Recognize that when two numbers are graphed by points on a number line, the point on the left corresponds to the lesser number.

B. Talk about and model the relationship between a number and its opposite, e.g., +5 is opposite -5, and -5 is opposite +5; a number plus its opposite is zero (additive inverse).

Stage III

A. Use a number line to initiate the development of the concept of absolute value.

B. Represent absolute value symbolically (notational form), if appropriate.

- OPERATIONS OF INTEGERS

Stage I

A. Model and develop basic rules of operations by considering, for example, positive and negative charges in a jar. Putting charges into the jar is addition, and removing charges from the jar is subtraction.

Stage II

A. Solve simple problems such as those involving money, operations on the number line, temperature, and football (yardage gained or lost) using models and symbols.

Stage III

A. Investigate the way positive and negative numbers can be handled on a calculator, e.g., Does the calculator have a +/- key or a + and - memory?

B. Solve problems in Stage II using the calculator.

- SOLVE SIMPLE EQUATIONS AND INEQUALITIES

Stage I

A. Develop the concept of ordered pairs through games and activities such as "Hurkel" and "Battleship."

Stage II

A. Use horizontal and vertical number lines to develop the "x" and "y" coordinate system.

B. Extend the ordered pairs concept to the four quadrants.

C. Graph the solution.

Stage III

A. Use "x" and "y" as representations for variables to solve and graph simple linear equations.

OVERRIDING THEMES

Problem Solving. Examples of themes that use signed numbers that can easily be used for problem solving activities are: deposits and withdrawals in banking, temperatures, yards gained and lost in football, distance above and below sea level, credits and debits, keeping score (in the hole and out of the hole), golf scores above and below par.

Estimation and Reasonableness of Results. When graphing linear equations, there is a built-in procedure for checking the reasonableness of results if the third point graphed does not lie on the same line as the other two.

Mental Computation. Computation with integers should either be done mentally or on the calculator -- depending on the size of the integers. Chains of mental computation exercises with integers are appropriate activities, e.g., $4 \cdot (-3) + (-7) + 5 =$, and patterns of integers can be done mentally, e.g., -2, -7, -11, __, __, . . .

Calculator/Computer Use. The change sign key on a calculator should be used with integer operations. There are several appropriate computer games (such as "Hurkle," "Battleship," and "Green Globs") which use signed numbers.

INSTRUCTIONAL MATERIALS

Calculators

Computers

Games and Software

Graph Paper

Grids

Maps or Charts

Number Lines

Straight Edge

Thermometers

9-12 MATHEMATICS CURRICULUM

OVERVIEW OF THE HIGH SCHOOL CURRICULUM MODEL

The curriculum model proposed here seeks to improve student preparation for vocations and further education. It has not tried to reach broad agreement by describing a traditional, conservative curriculum.

> Many mathematics educators working with emerging electronic technology see neither stability nor consensus in the future of school mathematics. They see, instead, striking opportunities and challenges to reshape content and pedagogy and the need for extensive research, development, and field testing of new programs.[11]

Problem solving, the study of patterns, and communication using mathematics should permeate all courses and topics in mathematics. Facts and techniques isolated from the overall structure of mathematics and its applications are of limited value and may alienate students.

> Mathematics has its own vocabulary, notation, and procedures. Just as working with integers is a learned skill, so are the abilities to listen, problem solve, locate information, and take a test. We need to take the responsibility for teaching math study skills because they are critical to our students' success in learning.[12]

Moreover, the curriculum must reflect the needs of students who will live in a highly technical society.

Problem solving is the first of 10 basic mathematical skills listed by the National Council of Supervisors of Mathematics (see appendix). Other basic skills include applying mathematics to everyday situations, alertness to the reasonableness of results, estimation and approximation, and appropriate computational skills including mental computation. This curriculum model is designed to enhance the learning of these basic skills.

Problems should arise from everyday situations. Numbers in the problems should have meaning. When numbers are not processed by pencil and paper, students are less distracted by the numerical computation and are able to keep more of the meaning in mind. Students must see reasonableness of results, estimation and approximation, and mental computation as important skills to learn. Requiring students to use pencil and paper to do computations that are more appropriate for a calculator may interfere with their mastery of these skills.

The committee wishes to emphasize their work on the curriculum model is in no sense a completed task. It is presented as a list of suggestions which should be a valuable aid to local schools as they study and build their own mathematics curriculum. This should be considered a working guide which will continue to evolve as the creative minds of the mathematics teachers of Iowa add, delete, and revise materials to fit the needs of their students.

All students should be encouraged to take at least three years of mathematics in high school. If possible, at least one mathematics course should be taken in the senior year. This will help ensure that the students' options for a variety of careers and courses of study remain open.

The following list of courses and topics combines traditional offerings that have served students well with some changes and additions that reflect the committee's view of students' needs in the near future. The suggested trigonometry course is the traditional one-semester course enhanced by the use of the calculator or computer.

In the description for the geometry course, selected questions are raised to help district personnel decide whether to offer one-semester or an entire year of geometry with the emphasis on proof being delayed until second semester. The assumption is that the presentation and content of K-8 mathematics includes a substantial commitment to developing geometric concepts and terminology. In the courses following geometry, specifically algebra II, trigonometry, and analytic geometry, logical development of ideas and careful use of language should be emphasized. Proof of results should be included whenever it is reasonably accessible and contributes to students' understanding. Deferring emphasis on proof in geometry has two advantages: (1) it allows the course to be shortened to one semester leaving time for discrete mathematics and/or statistics and (2) the students enrolled in algebra II, analytic geometry, and trigonometry are more mathematically mature and thus more able to understand and originate proofs of theorems.

The technical mathematics course is for students who need mathematics background for a vocational program. With the greater emphasis on proof in trigonometry and analytic geometry, the application orientation of the technical mathematics course provides a needed alternative.

Descriptive statistics is included as a part of the high school mathematics program since statistical data and reports are used by the media, and in many non-mathematical high school courses. If students are to become literate readers and listeners, it is an important communication skill. They need to have a rudimentary background — reading, interpreting, and understanding — to fully grasp the meaning of the data used in media reports and other content areas.

The general mathematics courses are not intended to be a rewrite of the previous work in the seventh and eighth grade. The topics were selected for their utility.

Outlined below is a sequence of courses for four high school population groups. The sequences are not rigid; on the contrary, students should be allowed the option of enrolling in any course for which they have the prerequisite. These populations are:

1. those who intend to stop their formal education with high school. The assumption is that they are typically less capable or less interested in mathematics, thus the emphasis on the use of practical applications and calculators.

2. those who intend to enter a technical skill field and will probably pursue further education in a one- or two-year program. No assumption is made that these students are less capable than others, rather they have different interests than those who intend to attain a degree in a four-year program.

3. those who intend to attain a B.A. in a liberal arts field. This sequence is based on the conviction that persons in any field must know many mathematical concepts and skills to cope in a society experiencing a technology explosion.

4. those who are mathematically talented. A five-year program is possible with algebra as an eighth grade course or concurrent enrollment in more than one mathematics course. If acceleration is desired, calculus could be the "fifth year" course. If calculus is the fifth year course, it must be a college equivalent course. If enrichment is desired, a list of topics is outlined under the heading "Fifth-Year Mathematics."

The best place to learn high school mathematics is in high school. Although many high school courses are offered at postsecondary institutions, such courses are often considered remedial. They are usually taught at an accelerated pace and the learning environment, such as class size and availability of individual help, is not always favorable. Moreover, some postsecondary institutions offer these remedial courses for a fee above the regular tuition and/or no college credit.

The topics and courses needed and available in grades 13 and 14 are not outlined as they range from mathematics for general education to specific courses designed to develop skills for a particular occupation. The committee suggests a separate study be made if curriculum suggestions are needed. Many of the topics and courses suggested for grades 9-12 could be, and are, included in some courses in grades 13 and 14.

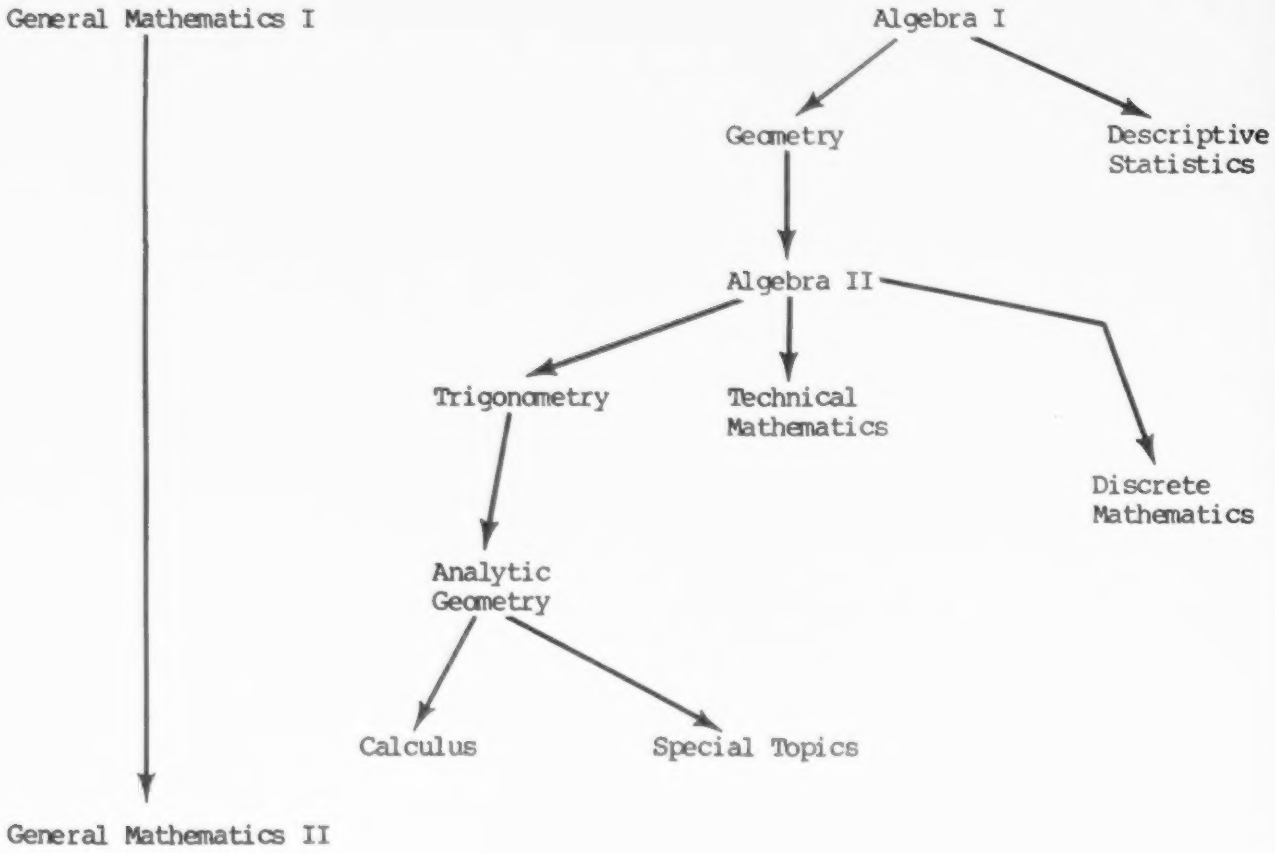
Computer science and mathematics are closely related (as are physics and mathematics), but they are not the same. The mathematics curriculum committee chose not to include computer science, computer programming, or data processing in the model. Computer science or computer programming credit should not be substituted for mathematics credit. The calculator and computer have tremendous potential as tools for learning, and it is highly recommended that they be integrated at every appropriate opportunity in the mathematics curriculum.

Table I
SUGGESTED STUDENT COURSE PLAN (9-12)

Student Population (Educational Expectation)	OPTION A		OPTION B		OPTION C	
	Grade	Course	Grade	Course	Grade	Course
High School Diploma	9	General Mathematics I	9	General Mathematics I		
	10	Algebra I (if capable)	10	Algebra I* (if capable)		
	11	—	11	Algebra I*		
	12	General Mathematics II	12	General Mathematics II		
2-Year Postsecondary Technical Program	9	Algebra I	9	Algebra I*	9	Algebra I
	10	Geometry and Algebra IIA	10	Algebra I*	10	Geometry
	11	Algebra IIB and Technical Mathematics	11	Geometry and Algebra IIA	11	Algebra IIA & IIB
	12	Descriptive Statistics (optional)	12	Algebra IIB and Technical Mathematics	12	Tech. Math & Descriptive Statis. (Opt.)
Baccalaureate - Liberal Arts				Algebra I		
	9	Algebra I	9	Geometry		
	10	Geometry and Algebra IIA	10	Algebra IIA and IIB		
	11	Algebra IIB and Descriptive Statistics	11	Descriptive Statistics, Discrete Mathematics, or Trigonometry		
	12	Discrete Mathematics and Trigonometry (optional)	12			
Baccalaureate - Mathematics, Science, Engineering			8	Algebra I		
	9	Algebra I	9	Geometry and Algebra IIA		
	10	Geometry and Algebra IIA	10	Algebra IIB and Trigonometry		
	11	Algebra IIB and Trigonometry	11	Analytic Geometry and Discrete Mathematics		
	12	Analytic Geometry - Discrete Mathematics	12	Calculus or Topics Course		

*Algebra I taught over a two-year period.

Figure 1
SUGGESTED HIGH SCHOOL COURSE SEQUENCE


General Mathematics I
General Mathematics II
Algebra I
Geometry
Descriptive Statistics
Algebra II
Trigonometry
Technical Mathematics
Discrete Mathematics
Analytic Geometry
Calculus
Special Topics

(Note: The arrows flow from course prerequisites.)

9 - 12 COURSES

Course Titles: ALGEBRA I, IIA, IIB

Description:

Variable is an important concept in algebra. Graphic representation provides a concrete image of the variable concept and should accompany algebraic manipulation throughout the courses. Manipulatives, applications, and problem solving should also be integrated throughout algebra, to provide meaning and a "real world" framework for the concepts learned.

More specific descriptions for Algebra I (two semester, sequential) and Algebra IIA and Algebra IIB are:

Course Title: ALGEBRA I (two semester, sequential)

Description:

Concepts are introduced and developed as far as time permits. Problem solving and graphic representation are emphasized along with manipulation. Proof is minimized.

Course Title: ALGEBRA IIA (one semester following the geometry course)

Description:

Basic algebraic concepts are reviewed and extended. Problem solving and graphic representation continue to be emphasized. Students' geometry background can be used to further their algebraic knowledge. Trigonometric functions can be introduced with primary emphasis on the concept of function and algebraic manipulation. Simple proofs are introduced.

Course Title: ALGEBRA IIB (one semester following Algebra IIA)

Description:

Students who reach this level are likely to be more serious and mature with respect to mathematics. Serious work with proof should begin in this semester when the proofs are accessible, and when proof will contribute to the understanding and use of the topics.

Content: (Note: The content for Algebra I and II are described as a four-semester program with an asterisk (*) indicating content which should be delayed until Algebra II.)

I. **The Language and Structure of Algebra**

   A. Set terminology

   B. The real number system and its subsystems

   C. Algebraic vocabulary: variable, constant, coefficient, exponent, absolute value, inverse

   D. Algebraic ground rules: order of operations, equivalence relations, field properties, algebraic proof

   E. Algebraic interpretation: words to symbols, symbols to words, problem solution

II. **Algebraic Expressions**

   A. Formulae

   B. Polynomials: simplification, operations, factoring

   C. Rational functions: simplification and operations

III. **Solutions and Applications of Equations and Inequalities**

   A. Linear: one and two variables

   B. Quadratic: factoring, completing the square, quadratic formula

   C. Radical

   D. *Complex number

   E. Systems of equations

IV. **Graphing**

   A. Linear: (includes slope, intercepts)

   B. Quadratic: (with rational roots)

   C. *Axes translations

V. **Radicals and Rational Exponents**

   A. Relationship

   B. Simplification

   C. Operations

VI. ***Complex Numbers**

   A. Relationship with radicals

- B. Simplification
- C. Arithmetic of operations

VII. Relations and Functions

- A. Notation
- B. Domain and range
- C. Graphs
- D. Direct and inverse variation
- E. *Algebra of functions
- F. *Exponential and logarithmic

VIII. Right Triangle Relationships

- A. Pythagorean Theorem
- B. Sine, cosine, tangent relations
- C. Applications

IX. *Matrices (with computers)

- A. Properties
- B. Matrix solution of systems

X. *Higher Order Polynomials

- A. Synthetic division: (Fundamental Theorem of Algebra, Remainder/Factor Theorem)
- B. Rational roots
- C. Gaussian Elimination

XI. *Discrete Algebra

- A. Sequences and series: arithmetic and geometric
- B. Mathematical induction
- C. Binomial Theorem and Pascal's Triangle
- D. Permutations and combinations
- E. Probability

Course Title: ANALYTIC GEOMETRY

Description:

Analytic geometry should be studied by students who intend to take calculus. It will provide the student insight into the unification of geometry and algebra concepts. Since proof may have been de-emphasized in the earlier geometry course, proof must be fundamental to this course. Analytic geometry provides an excellent opportunity to compare synthetic, analytic, and vector proofs of the same theorems. Calculators and computers should be used where appropriate.

Content:

I. Basic Concepts

   A. Linear equation in one variable

   B. Quadratic equations

   C. Determinants

   D. Intervals

II. Points and Vectors in Two Dimensions

   A. Rectangular coordinate system

   B. Symmetry

   C. Projections

   D. Scalar components of a segment

   E. Distance between points

   F. Vectors in the plane

   G. Angle between two vectors

   H. Parallel and perpendicular vectors

   I. The coordinates of a point that divides a segment in a given ratio

III. Points and Vectors in Three Dimensions

   A. The point in space

   B. Projections

C. Scalar components of a segment

D. Length or magnitude of a segment

E. Vectors in space

F. Cosine of the angle between two vectors

G. The coordinates of a point that divides a segment in a given ratio

IV. **The Straight Line**

A. Unit vectors

B. Translations

C. Equations of lines

1. General
2. Slope-intercept
3. Point-slope
4. Parametric
5. Intersecting lines

D. Angle between two lines

E. Parallel and perpendicular lines

F. Distance from a line to a point

V. **The Plane**

A. Equations of the plane

B. Parallel and perpendicular planes

C. Cross product

D. Determine a plane satisfying three conditions

E. Distance from a plane to a point

F. Distance between two planes

VI. **The Circle**

- A. Equations
    - 1. Standard
    - 2. General
    - 3. Parametric
- B. Circle determined by three conditions
- C. Tangent lines
- D. Radical axis

VII. **The Conics**

- A. Equations and graphs
    - 1. Parabola
    - 2. Ellipse
    - 3. Hyperbola
- B. Translations
- C. Points of Intersection

VIII. **Polar Coordinates**

- A. Coordinates of a point
- B. Relation between polar and rectangular coordinates
- C. Sketching curves

IX. **Graphing in Space**

- A. Lines and planes
- B. Quadric surfaces
- C. Cylindrical and spherical coordinates

Course Title: DESCRIPTIVE STATISTICS

Description:

The one-semester course is for students who have completed one year of algebra. The course is intended for students who may end their education with high school or students who intend to further their education in a non-mathematical field.

Throughout the course the students should be involved in hands-on applications of the content being studied. As has been emphasized in other courses, the careful use of the language of statistics should be stressed. Calculators and computers should be used where appropriate, with time set aside to instruct students on their proper use.

Content:

I. Collection of Data

  A. Randomness

  B. Bias

  C. Sample population

  D. Experimental design

    1. Random sampling

    2. Surveys

    3. Simulations

II. Preliminary Analysis of Data

  A. Stem and leaf

  B. Scatter plots

  C. Box and whisker

  D. Graphs

    1. Bar

    2. Line

    3. Pictographs

    4. Misleading graphs

  E. Misleading graphs

III. **Central Tendency**

  A. Median

  B. Mean ($\sum$ notation)

  C. Selecting proper averages

  D. Drawing erroneous conclusions from averages

IV. **Variability**

  A. Range

  B. Mean deviation

  C. Variance

  D. Standard deviation

V. **Basic Probability**

VI. **Normal Distribution**

  A. Standard scores

  B. Normal curve characteristics

  C. Using z-score tables

VII. **Central Limit Theorem** (using simulations)

VIII. **Correlation**

  A. Scatter plots

  B. Straight line equation

  C. Median fit line

  D. Correlation versus "cause and effect"

IX. **Regression**

  A. Using mathematics to predict

  B. Construct the "best fit" line

X. **Hypothesis Testing**

Course Title: DISCRETE MATHEMATICS

Description:

Designing or modeling complex switching circuits for a computer requires discrete mathematics; working out the orbits of space vehicles requires continuous mathematics. Discrete mathematics is more or less the same part of mathematics as what has been called finite mathematics. It is not calculus, which is part of continuous mathematics. Discrete mathematics includes logic, counting by permutations and combinations, probability models having only a finite number of outcomes, searching for best or optimal outcomes in scheduling complex activities, and finding the shortest routes in transportation network graphs.

Students whose educational/vocational goals do not depend on at least two semesters of calculus may find a course in discrete mathematics more useful than the traditional trigonometry/analytic geometry sequence.

This one-semester course rests on the skills, knowledge, and maturity of students with two years of algebra. It serves as a bridge to a rapidly growing area of applied mathematics, much as the pre-calculus courses in trigonometry and analytic geometry serve as a bridge to classical applied mathematics, which begins with calculus. The applications of discrete mathematics are somewhat different than those in traditional continuous mathematics, but are just as significant and deep. While the content of an introductory course in discrete mathematics is not yet well established, most courses will be organized around the major themes: recursion/algorithm/induction, sets/counting/induction, logic/Boolean algebra, algorithmic linear algebra, graphs/trees, and statistics. As in other courses, we urge the careful use of language and the inclusion of proofs when they are accessible and contribute directly to understanding. Many of the topics in this course are better understood and learned with appropriate computer support.

Content:

I. Recurrence/Difference Equations

   A. Population growth

   B. Simple and compound interest

   C. Amortization

   D. Annuities

   E. Installment buying

   F. Mathematical induction

II. **Sets and Counting Techniques**

   A. Permutations

   B. Combinations

   C. Binomial coefficients

   D Counting elements in sets formed from other sets

   E. Pigeon-hole principle

III. **Finite Probability**

IV. **Random Variables**

   A. Expectation

   B. Binomial distribution

   C. Normal distribution (as a means to analyze large binomial populations)

V. **Algorithmic Linear Algebra**

   A. Matrix operations

   B. Relations to graphs

   C. Row operations

   D. Gauss-Jordan algorithm

   E. Algorithm for matrix inverse and use in solving $A\vec{x} = \vec{b}$

   F. Linear programming

   G. Markov chains

VI. **Graphs** (modeling with graphs and algorithms for solving search problems on flow problems)

VII. **Statistics**

   A. Sampling

   B. Descriptive statistics

   C. Hypothesis testing

A SELECTED LIST OF TEXTBOOKS IN DISCRETE STRUCTURES

Doerr, Alan, and Kenneth Levasseur. Applied Discrete Structures for Computer Science. (ISBN 0-574-21755-X). Chicago: Science Research Associates, 1985.

Fisher, James L. Application-Oriented Algebra: An Introduction to Discrete Mathematics. (ISBN 0-7002-25404-8, HarpC). New York: Harper and Row, 1977.

Gersting, Judith L. Mathematical Structures for Computer Science. (ISBN 0-7167-1305-5). New York: W. H. Freeman, 1982.

Grimaldi, Ralph P., and Rose-Hulman. Discrete & Combinatorial Mathematics. (ISBN 0-201-12590-0). Reading, MA: Addison-Wesley, 1985.

Johnsonbaugh, Richard. Discrete Mathematics. (ISBN 0-02-360900-1). New York: Macmillan, 1984.

Kolman, Bernard, and Robert Beck. Elementary Linear Programming and Applications (Computer Science and Applied Mathematics Series). Text Edition (ISBN 0-12-417860-X); Instrs' Manual (ISBN 0-12-417862-6). Orlando, FL: Academic Pr., Inc., 1980.

Kolman, Bernard, and Robert C. Busby. Discrete Mathematical Structures for Computer Science. (ISBN 0-13-215418-8). Englewood Cliffs, NJ: Prentice-Hall, 1984.

Liu, C. L. Elements of Discrete Mathematics. (ISBN 0-07-038133-X). New York: McGraw-Hill, 1985.

Marcus, Marvin. Discrete Mathematics: A Computational Approach Using BASIC. Paper Edition (ISBN 0-914894-38-2); Apple diskette (ISBN 0-88175-001-8); IBM diskette (ISBN 0-88175-091-3). Rockville, MD: Computer Science Press, 1983.

Norris, Fletcher R. Discrete Structures: An Introduction to Mathematics for Computer Scientists. (ISBN 0-13-215260-6). Englewood Cliffs, NJ: Prentice-Hall, 1985.

Polimeni, Albert D., and H. Joseph Straight. Foundations of Discrete Mathematics (Mathematics Series). (ISBN 0-534-03612-0). Monterey, CA: Brooks/Cole, 1985.

Ross, Kenneth A., and Charles R. Wright. Discrete Mathematics. (ISBN 0-13-215286-X). Englewood Cliffs, NJ: Prentice-Hall, 1985.

Stanat, Donald F., and David F. McAlister. Discrete Mathematics in Computer Science. (ISBN 0-13-216150-8). Englewood Cliffs, NJ: Prentice-Hall, 1977.

Tremblay, J. P., and R. Manohar. Discrete Mathematical Structures with Applications to Computer Science (Computer Science Series). (ISBN 0-07-065142-6). New York: McGraw-Hill, 1975.

Tucker, Alan. Applied Combinatorics. (ISBN 0-471-86371-8). New York: John Wiley and Sons, 1984.

Course Title: **FIFTH-YEAR MATHEMATICS**

Description:

For those mathematically talented students who would like a fifth year of college preparatory mathematics, two options are recommended.

Content:

I. **Option 1** — A one-year course of college level calculus.

II. **Option 2** — A topics course. This course could take several approaches: (1) a few topics covered in depth or (2) several topics studied to provide a broad perspective of the field of mathematics. The topics would be composed of selected units from the following:

   A. Linear algebra

   B. Probability

   C. Number theory

   D. Inferential statistics

   E. Non-Euclidean geometry

   F. Solid geometry

   G. Projective geometry

   H. Mathematical modeling

   I. Mathematical applications (using computers)

   J. Mathematics of finance

   K. Topology

   L. Boolean algebra

Course Title: GENERAL MATHEMATICS I (Survey of Mathematics)

Description:

This survey is a one-year course for students who are undecided or not prepared for the first year of algebra. It provides techniques in how to study mathematics and how to deal with mathematical anxiety. Students should not use paper and pencil to perform addition and subtraction on numbers larger than five digits. Factors should not exceed three digits. Divisors should be limited to two digits and dividends to four digits. Problem solving and calculator usage should permeate the course. An introduction of algebra is included near the end of the course to provide background for students who may wish to enroll in algebra as their next mathematics course. (Note: Teachers should have and use an overhead calculator.)

Content:

I. **How to Study Mathematics**

   A. Language of mathematics

   B. Reading mathematics

   C. Listening skills

   D. Taking notes

   E. Studying and taking tests

   F. Dealing with mathematics anxiety

II. **Calculator Usage**

   A. How to use

   B. When to use

   C. Reasonableness of results

III. **Problem-Solving Strategies**

   A. Guess and check

   B. Look for a pattern

   C. Make a systematic list

   D. Make and use a drawing or model

   E. Eliminate possibilities

IV. **Geometry**

- A. Recognize and name two- and three-dimensional geometric figures.
- B. Constructions
- C. Congruence and similarity
  - 1. direct measurement
  - 2. indirect measurement
- D. Slides, flips, and turns (translations, reflections, rotations)
- E. Graphing
- F. Coordinate geometry
- G. String art/curve stitching

V. **Measurement Including Estimation, Precision, and Accuracy for Both Metric and Customary Units** (Use actual measuring devices during learning activities.)

- A. Linear (including perimeter and circumference)
- B. Area
- C. Volume
- D. Surface area
- E. Mass (weight)
- F. Time
- G. Temperature

VI. **Basic Facts and Computing With Multiples of Powers of Tens**

VII. **Estimation and Mental Arithmetic**

- A. Rounding
- B. Truncating
- C. Range of answer

VIII. **Whole Numbers** (use powers of ten blocks -- depending on the students' past experience)

- A. Basic operations (+, -, x, -)
- B. One- and multi-step word problems

IX. **Fractions** (use fraction bars — depending on students' previous experience)

- A. Least common denominator
- B. Equivalent expressions
- C. Mixed and improper
- D. Basic operations (only enough instructional time devoted to these for success in subsequent topics)
- E. One- and multi-step word problems

X. **Decimals** (use powers-of-ten blocks, decimal squares — depending on students' previous experience)

- A. Place value
- B. Basic operations
- C. Relationship between fractions and decimals
- D. One- and multi-step word problems
- E. Applications - money

XI. Ratio, Proportion, and Percent

- A. Equivalent ratios
- B. Solving proportions
- C. Relation between ratio, decimal, fractions, and percent
- D. One- and multi-step word proble
- E. Applications - money

XII. Introduction to Algebra

- A. Integers
- B. Translations (words to symbols and symbols to words)
- C. Linear equations
- D. Simple word problems

Course Title: GENERAL MATHEMATICS II

Description:

The second year of general mathematics is intended for students in their senior year. The rationale for teaching the second year of general mathematics at this time is that students will be more mature and be more serious about mathematics and its applications. Problem solving, applications, and calculator usage should permeate the course.

Content (first semester):

I. Transportation
    A. Automobile Ownership
        1. Car loans
            a. Compare annual percentage rates (APR) at small loan companies, credit unions, banks, and car dealers
            b. Amount of payment using monthly payment tables or calculators
        2. Automobile Insurance
            a. Coverage from property damage, bodily-injury, collision, comprehensive, and medical payments insurance
            b. Iowa Financial Responsibilities Law
        3. Operating cost for gas and oil
    B. Map reading
    C. Using the distance formula
II. Consumerism
    A. Comparison of unit cost
    B. Understanding warranties
    C. Counting change
    D. Determining sales tax
    E. Computing the discount on a purchase

- F. Determining the rate of discount on a purchase
- G. Revolving charge accounts and credit cards
- H. Gratuities

III. Income

- A. Determining earnings on hourly basis and commission basis
- B. Determining overtime earnings
- C. Payroll deductions
    1. Income tax withholdings
    2. Social Security withholding
    3. Miscellaneous deductions (health insurance, annuities, savings bonds, . . .)

IV. Taxes

- A. State forms
- B. Federal forms

V. Banking

- A. Opening a checking or savings account
- B. Computing interest
- C. Computing the growth of regular deposits
- D. Determining service charges on checking accounts
- E. Reconciling the checking account statement

VI. Investments (optional)

- A. How to read the stock market report
- B. Use net change
- C. Compute dividends on stock
- D. Compute rate of return on an investment
- E. Mutual funds
- F. How to read the bond market report

G. Compute interest on bonds

H. Government and municipal bonds

Content (second semester):

I. **Read Statistical Displays**

   A. Interpret information from tables, charts, and graphs as presented in newspapers and magazines

   B. Evaluate and draw conclusions from tables, charts, and graphs

   C. Evaluate sampling procedures (questionnaires, surveys)

II. **Collect and Organize Data Into Tables, Charts, and Graphs**

   A. Identify methods for collecting data

   B. News clippings, surveys, experiments as sources of data

   C. Make frequency distributions and scatter diagram

   D. Organize data into tables, charts, and graphs (including stem and leaf plots, box and whisker plots)

III. **Summarize Data**

   A. Use central tendency and dispersion measures (including standard deviation)

   B. Identify different uses of numbers in statistical reporting

   C. Use statistics in decision making

IV. **Statistics in Society**

   A. Recognize the importance of statistics

   B. Identify misuses of statistics (such as misleading graphs and the use of an inappropriate measure of central tendency)

V. **Probability**

   A. Events (mutually exclusive and independent)

   B. Sample spaces

   C. Counting techniques

   D. Permutations

E. Combinations

F. Simple and compound probability

G. Conditional probability

H. Odds

Course Title: GEOMETRY

Description:

The curriculum model attempts to incorporate current learning theory into the geometry classroom. A process of skill development in geometry has been identified in the van Hiele model. This model is described as follows:

Level 1	Recognition - learn some vocabulary and recognize a shape as a whole.
Level 2	Analysis - analyze properties of figures
Level 3	Ordering - logically order geometric figures (all squares are rectangles, but not all rectangles are squares) and understand interrelationships between figures and the importance of accurate definitions. Deductive thinking begins to take shape as the students' collection of visual properties of various shapes ask for organization.
Level 4	Deduction - understand the significance of deduction and the role of postulates, theorems, and proof. Write original proofs.
Level 5	Rigor - understand the importance of precision in dealing with foundations, such as the collection of axioms and postulates at the foundation of Euclidean geometry.[13]

The proposed K-8 curriculum suggests the inclusion of many topics and activities in geometry not typically included in most K-8 sequences. Following these suggestions should lead the students through van Hiele's Levels 1 and 2. The next step is the high school geometry course. The main focus of this course has been to teach the axiomatic structure of mathematics and to teach proof. This goal has not been accomplished for the majority of students. This leads one to raise two questions: (1) Should proof be taught as part of the geometry course? and (2) If proof is taught, how should it be done?

The answer to the first question is determined by the student population enrolled in geometry. The majority of these students do not need an emphasis on proof. The development of proofs by students is better left to subsequent courses where the majority of students are more mathematically mature and more mathematically oriented.

The change of emphasis in the geometry course to exclude writing formal proofs makes it possible to teach the course in one semester. In adopting this idea for a geometry course, schools must revise subsequent courses to include more emphasis on proof.

A goal of this suggested one-semester course is to develop geometric concepts and relationships and to enhance the students' ability to follow (not write) mathematical arguments and to discern flaws in arguments. Concepts in geometry should be developed concretely in the beginning of the course (e.g., measure triangles as a test for congruence). The correct use of mathematics language, logical development of concepts, and measurement (direct and indirect) must be emphasized throughout the course. Symmetry, similarity, and congruence are unifying concepts to be developed, expanded, and extended throughout the course. As the course develops, the activities should become more abstract. Students also need to be able to visualize and draw two- and three-dimensional figures. This should lead the students through van Hiele Level 3.

If the philosophy is to teach proof in the geometry course, then the second question must be answered — How should proof be taught? Proof is at van Hiele's Level 4. To reach this level the students must pass through Levels 1, 2, and 3. It is suggested that the proposed K-8 curriculum and one-semester geometry course be taught to take the students through Level 3. For students who are capable and need to reach Level 4 (and possibly Level 5), a second semester of geometry emphasizing the axiomatic structure of Euclidean geometry could be offered.

The second-semester geometry course should follow the traditional development of Euclidean geometry, possibly beginning with congruence of triangles (Item VI of the outline of topics). It should emphasize the significance of deduction and the role of definitions, postulates, theorems, and proof. When writing proofs, a variety of methods and styles should be used. As the course develops, the traditional two-column proof should be used to develop proof in the paragraph form. Students should be able to recognize invalid as well as valid arguments. Time should also be spent studying non-Euclidean geometries to reinforce the concept and importance of postulates and definitions.

Content:

I. Introduction

  A. Language

  B. Notation

  C. Drawing and sketching diagrams

II. Classification of Two- and Three-Dimensional Figures

  A. Angles

  B. Angle pairs

  C. Triangles

D. Polygons

E. Solids

III. Constructions and Loci

A. Compass and straight edge

B. Miras

IV. Parallelism in Two and Three Dimensions

V. Perpendicularity in Two and Three Dimensions

VI. Congruent Triangles

VII. Polygons

VIII. Similar Polygons

IX. Right Triangles

X. Circles

XI. Measurement: Direct and Indirect

A. Perimeter and circumference

B. Area and surface area

C. Volume

XII. Transformations

A. Translations

B. Rotations

C. Reflections

XIII. Coordinate Geometry

Course Title: **TECHNICAL MATHEMATICS**

Description:

Technical mathematics is a one-semester course for students who have completed Algebra II. The course is intended for students who plan to further their education in a postsecondary vocational program. This does not preclude other students from taking the course. The course should emphasize applications and the use of technology. Scientific calculators and computers should be used where appropriate.

Content:

I. **Measurement**
    A. Precision and accuracy
    B. Linear
    C. Area and volume
        1. English units
        2. Metric units
    D. Angular

II. **Geometry**
    A. Lines
    B. Angles
    C. Triangles
    D. Parallel lines and angle relationships
    E. Congruence and similarity of triangles
    F. Right triangles
        1. Special right triangles (30-60 and 45-45)
        2. Trigonometry

III. **Solutions to Systems of Equations**
    A. Graphing
    B. Algebraic

C. Matrices

1. Operations

2. Inverse

D. Gaussian elimination

IV. **Ratio, Proportion, Percent**

V. **Direct, Inverse, Joint Variation**

VI. **Relations and Functions** (properties and graphs)

A. Linear

B. Quadratic

C. Logrithmic and exponential

D. Trigonometric (unit circle)

VII. **Vectors**

A. Properties

B. Addition

1. Graphically

2. Analytically (laws of sine and cosine)

VIII. **Velocity and Acceleration**

IX. Analytic Geometry

A. Distance

1. Two point

2. Point and line

3. Two lines

B. Conic Sections

C. Polar Coordinates

D. Curve Sketching

E. Complex Numbers

Course Title: TRIGONOMETRY

Description:

The unifying concepts in trigonometry are functions, graphing, and proof. A functional approach integrating as many concepts as possible should be used throughout the course. Proofs of the trigonometric relations and identities are essential. Basic definitions, basic identities, and reference angles should be based on the unit circle. Graphical characteristics of relations and functions, especially the trigonometric functions, and their inverses should permeate the course. Tables, particularly interpolation in tables, should be de-emphasized. Calculations should be done with scientific calculators. Computers can be used effectively where appropriate. (Note: Students may have had some experience with right triangle trigonometry in previous courses.)

Content:

I. **Properties of Algebraic and Trigonometric Functions**

   A. Ordered pairs

   B. Domain and range

   C. Wrapping (around squares, triangles, hexagons, etc.)

   D. Periodic

   E. Translations

   F. Sum and difference

   G. Composite

   H. Inverse

II. **Circular Functions**

   A. Central angle and arc length

   B. Radians and degrees

   C. Sine, cosine, and tangent (functions)

   D. Reference angles

   E. Pythagorean identities

   F. Secant, cosecant, cotangent

   G. Basic identities

III. Identities and Equations Involving Trigonometric Functions

- A. Co-functions
- B. Reduction formulas
- C. Sum and difference of angles (e.g., [$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$])
- D. Sum and product formulas
- E. Double and half-angle formulas
- F. Solution of equations

IV. Measurement and Applications Using the Trigonometric Functions

- A. Values of special angles
- B. Reference angles
- C. Graphing basic trigonometric functions
- D. Graphing by adding ordinates
- E. Effect of parameters (amplitude, period, phase shift)
- F. Solution of right triangles
- G. Indirect measurement
- H. Law of sines
- I. Law of cosines
- J. Vectors (forces, velocities, etc.)

V. Enrichment Topics

- A. Complex numbers
- B. Complex plane
- C. Polar form
- D. DeMoivre's Theorem
- E. $e^{i\theta} = \text{Cos } \theta + i \sin \theta$
- F. $A \sin \theta + B \cos \theta = \sqrt{A^2 + B^2} \sin(\theta + \alpha)$

  where $\sin \alpha = \dfrac{B}{\sqrt{A^2 + B^2}}$ and $\cos \alpha = \dfrac{A}{\sqrt{A^2 + B^2}}$

Evaluation

EVALUATING STUDENT PROGRESS

Evaluation is not limited to testing. It includes gathering data and interpreting the data. Testing is one source of data. There are many others. Today, many people use test scores as the sole index of the quality of mathematics programs or of the success of student achievement. Test scores alone should not be considered synonymous with achievement or program quality. A serious danger to the education of our youth is the increasing tendency on the part of the public to assume that the sole objective of schooling is a high test score. This is often assumed without the critical knowledge of what is being tested or whether test items fit desired goals.

The evaluation of problem-solving performance will demand new approaches to measuring. Certainly present tests are not adequate. In particular, the measuring of the use of problem-solving processes will demand innovative techniques. Evaluation of programs with program-solving goals must be sensitive to the nature of those goals.[14]

To make effective instruction decisions, teachers need to know to what extent students understand mathematical concepts, the level of their skill development, and the thinking strategies they employ in problem solving. Teachers need to constantly monitor students' mathematics language, thinking processes, and written work to obtain adequate feedback for making instructional decisions.

Some techniques for evaluating student progress include:

1. observing
2. listening
3. demonstrating (Student show how to use place value blocks to compute a sum.)
4. questioning
5. interviewing
6. homework (Students do comparison shopping using newspaper advertisements or visiting stores.)
7. quizzes
8. testing
9. board work
10. class discussions

The mathematical content and instructional and learning strategies previously described in this guide do not lend themselves only to paper and pencil measures. In mathematics curriculum planning, consideration must be given to evaluating the students' progress in attaining a knowledge base and solving problems. This progress can best be determined by using a variety of techniques.

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Appendices

APPENDIX

NATIONAL COUNCIL OF SUPERVISORS OF MATHEMATICS
POSITION PAPER ON BASIC MATHEMATICAL SKILLS

Introduction

The currently popular slogan "Back to the Basics" has become a rallying cry of many who perceive a need for certain changes in education. The result is a trend that has gained considerable momentum and has initiated demands for programs and evaluations which emphasize narrowly defined skills.

Mathematics educators find themselves under considerable pressure from boards of education, legislatures, and citizens' groups who are demanding instructional programs which will guarantee acquisition of computational skills. Leaders in mathematics education have expressed a need for clarifying what are the basic skills needed by students who hope to participate successfully in adult society.

The narrow definition of basic skills which equates mathematical competence with computational ability has evolved as a result of several forces:

1. Declining scores on standardized achievement tests and college entrance examinations;
2. Reactions to the results of the National Assessment of Educational Progress;
3. Rising costs of education and increasing demands for accountability;
4. Shifting emphasis in mathematics education from curriculum content to instructional methods and alternatives;
5. Increased awareness of the need to provide remedial and compensatory programs;
6. The widespread publicity given to each of the above by the media.

This widespread publicity, in particular, has generated a call for action from governmental agencies, educational organizations, and community groups. In responding to these calls, the National Institute of Education adopted the area of basic skills as a major priority. This resulted in a Conference on Basic Mathematical Skills and Learning, held in Euclid, Ohio, in October, 1975.

The National Council of Supervisors of Mathematics (NCSM), during the 1976 Annual Meeting in Atlanta, Georgia, met in a special session to discuss the Euclid Conference Report. More than 100 members participating in that session expressed the need for a unified position on basic mathematical skills which would enable them to provide more effective leadership within their respective school systems, to give adequate rationale and direction in their tasks of implementing basic mathematics programs, and to appropriately expand the definition of basic skills. Hence, by an overwhelming majority, they mandated the NCSM to establish a task force to formulate a position on basic mathematical skills. This statement is the result of that effort.

Rationale for the Expanded Definition

There are many reasons why basic skills must include more than computation. The present technological society requires daily use of such skills as estimating, problem solving, interpreting data, organizing data, measuring, predicting, and applying mathematics to everyday situations. The changing needs of society, the explosion of the amount of quantitative data, and the availability of computers and calculators demand a redefining of the priorities for basic mathematics skills. In recognition of the inadequacy of computation alone, NCSM is going on record as providing both a general list of basic mathematical skills and a clarification of the need for such an expanded definition of basic skills.

Any list of basic skills must include computation. However, the role of computational skills in mathematics must be seen in the light of the contributions they make to one's ability to use mathematics in everyday living. In isolation, computational skills contribute little to one's ability to participate in mainstream society. Combined effectively with the other skills areas, they provide the learner with the basic mathematical ability needed by adults.

Defining Basic Skills

The NCSM views basic mathematical skills as falling under ten vital areas. The ten skill areas are interrelated and many overlap with each other and with other disciplines. All are basic to pupils' development of the ability to reason effectively in varied situations.

This expanded list is presented with the conviction that mathematics education must not emphasize computational skills to the neglect of other critical areas of mathematics. The ten components of basic mathematical skills are listed below, but the order of their listing should not be interpreted as indicating either a priority of importance or a sequence for teaching and learning.

Furthermore, as society changes our ideas about which skills are basic also change. For example, today our students should learn to measure in both the customary and metric systems, but in the future the significance of the customary system will be mostly historical. There will also be increasing emphasis on when and how to use hand-held calculators and other electronic devices in mathematics.

Ten Basic Skill Areas

1. **Problem Solving** - Learning to solve problems is the principal reason for studying mathematics. Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations. Solving word problems in texts is one form of problem solving, but students also should be faced with non-textbook problems. Problem-solving strategies involve posing questions, analyzing situations, translating results, illustrating results, drawing diagrams, and using trial and error. In solving problems,

students need to be able to apply the rules of logic necessary to arrive at valid conclusions. They must be able to determine which facts are relevant. They should be unfearful of arriving at tentative conclusions and they must be willing to subject these conclusions to scrutiny.

2. **Applying Mathematics to Everyday Situations** - The use of mathematics is interrelated with all computation activities. Students should be encouraged to take everyday situations, translate them into mathematical expressions, solve the mathematics, and interpret the results in light of the initial situation.

3. **Alertness to the Reasonableness of Results** - Due to arithmetic errors or other mistakes, results of mathematical work are sometimes wrong. Students should learn to inspect all results and to check for reasonableness in terms of the original problem. With the increase in the use of calculating devices in society, this skill is essential.

4. **Estimation and Approximation** - Students should be able to carry out rapid approximate calculations by first rounding off numbers. They should acquire some simple techniques for estimating quantity, length, distance, weight, etc. It is also necessary to decide when a particular result is precise enough for the purpose at hand.

5. **Appropriate Computational Skills** - Students should gain facility with addition, subtraction, multiplication, and division with whole numbers and decimals. Today it must be recognized that long, complicated computations will usually be done with a calculator. Knowledge of single digit number facts is essential and mental arithmetic is a valuable skill. Moreover, there are everyday situations which demand recognition of, and simple computation with, common fractions.

   Because consumers continually deal with many situations that involve percentage, the ability to recognize and use percents should be developed and maintained.

6. **Geometry** - Students should learn the geometric concepts they will need to function effectively in the 3-dimensional world. They should have knowledge of concepts such as point, line, plane, parallel, and perpendicular. They should know basic properties of simple geometric figures, particularly those properties which relate to measurement and problem-solving skills. They also must be able to recognize similarities and differences among objects.

7. **Measurement** - As a minimum skill, students should be able to measure distance, weight, time, capacity, and temperature. Measurement of angles and calculations of simple areas and volumes are also essential. Students should be able to perform measurement in both metric and customary systems using the appropriate tools.

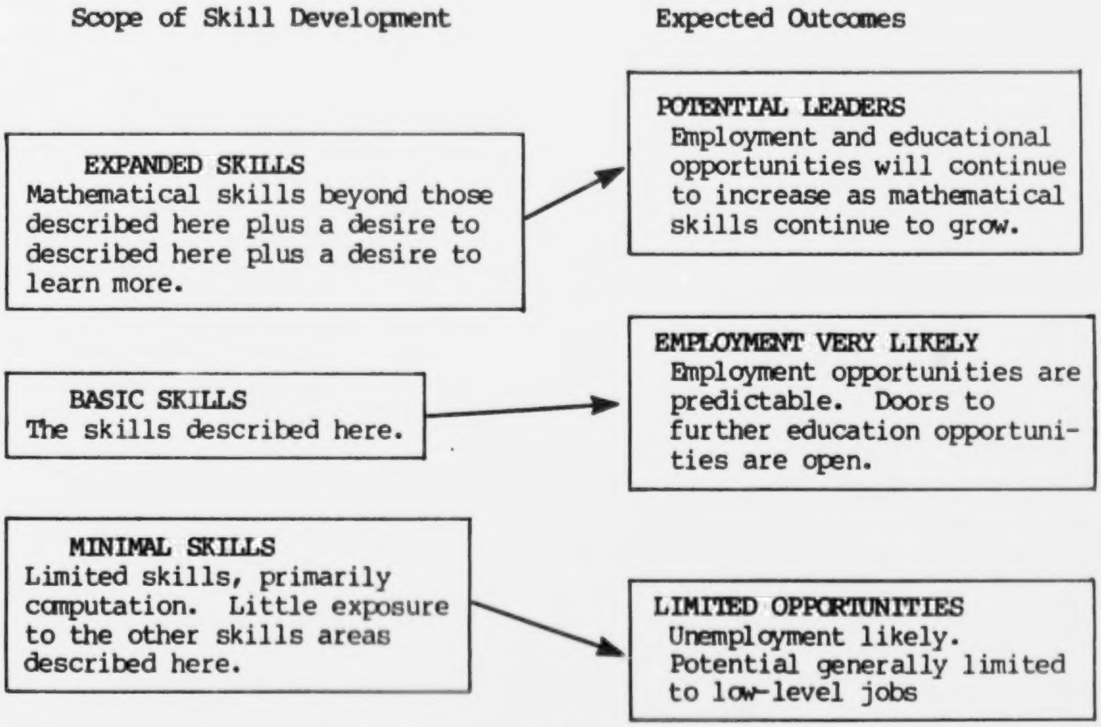
8. **Reading, Interpreting, and Constructing Tables, Charts, and Graphs** - Students should know how to read and draw conclusions from simple tables, maps, charts, and graphs. They should be able to condense numerical information into more manageable or meaningful terms by setting up simple tables, charts, and graphs.

9. **Using Mathematics to Predict** - Students should learn how elementary notions of probability are used to determine the likelihood of future events. They should learn to identify situations where immediate past experience does not affect the likelihood of future events. They should become familiar with how mathematics is used to help make predictions such as election forecasts.

10. **Computer Literacy** - It is important for all citizens to understand what computers can and cannot do. Students should be aware of the many uses of computers in society, such as their use in teaching/learning, financial transactions, and information storage and retrieval. The "mystique" surrounding computers is disturbing and can put persons with no understanding of computers at a disadvantage. The increasing use of computers by government, industry, and business demands an awareness of computer uses and limitations.

Basic Skills and the Student's Future

Anyone adopting a definition of basic skills should consider the "door-opening/door-closing" implications of the list. The following diagram illustrates expected outcomes associated with various amounts of skill development.



Scope of Skill Development

Expected Outcomes

EXPANDED SKILLS
Mathematical skills beyond those described here plus a desire to described here plus a desire to learn more.

POTENTIAL LEADERS
Employment and educational opportunities will continue to increase as mathematical skills continue to grow.

BASIC SKILLS
The skills described here.

EMPLOYMENT VERY LIKELY
Employment opportunities are predictable. Doors to further education opportunities are open.

MINIMAL SKILLS
Limited skills, primarily computation. Little exposure to the other skills areas described here.

LIMITED OPPORTUNITIES
Unemployment likely. Potential generally limited to low-level jobs

Minimum Essentials for High-School Graduation

Today some school boards and state legislatures are starting to mandate mastery of minimum essential skills in reading and mathematics as a requirement for high-school graduation. In the process, they should consider the potential pitfalls of doing this without an appropriate definition of "basic skills." If the mathematics requirements are set inordinately high, then a significant number of students may not be able to graduate. On the other hand, if the mathematics requirements are set too low and mathematical skills are too narrowly defined, the result could be a sterile mathematics program concentrating exclusively on learning of low-level mathematical skills. This position paper neither recommends nor condemns minimal competencies for high-school graduation. However, the ten components of basic skills stated here can serve as guidelines for state and local school systems that are considering the establishment of minimum essential graduation requirements.

Developing the Basic Skills

One individual difference among students is style or way of learning. In offering opportunities to learn the basic skills, options must be provided to meet these varying learning styles. The present "back-to-basics" movement may lead to an emphasis on drill and practice as a way to learn.

Certainly drill and practice is a viable option, but it is only one of many possible ways to bring about learning and to create interest and motivation in students. Learning centers, contracts, tutorial sessions, individual and small-group projects, games, simulations and community-based activities are some of the other options that can provide the opportunity to learn basic skills. Furthermore, to help students fully understand basic mathematical concepts, teachers should utilize the full range of activities and materials available, including objects the students can actually handle.

The learning of basic mathematical skills is a continuing process with extends through all of the years a student is in school. In particular, a tendency to emphasize computation while neglecting the other nine skill areas at the elementary level must be avoided.

Evaluating and Reporting Student Progress

Any systematic attempt to develop basic skills must necessarily be concerned with evaluating and reporting pupil progress.

In evaluation, test results are used to judge the effectiveness of the instructional process and to make needed adjustments in the curriculum and instruction for the individual student. In general, both educators and the public have accepted and emphasized an overuse of and overconfidence in the results of standardized tests. Standardized tests yield comparisons between students and can provide a rank ordering of individuals, schools, or districts. However, standardized tests have several limitations including the following:

a. Items are not necessarily generated to measure a specific objective or instructional aim.
b. The tests measure only a sample of the content that makes up a program; certain outcomes are not measured at all.

Because they do not supply sufficient information about how much mathematics a student knows, standardized tests are not the best instruments available for reporting individual pupil growth. Other alternatives such as criterion tests or competency tests must be considered. In criterion tests, items are generated which measure the specific objectives of the program and which establish the student's level of mastery of these objectives. Competency tests are designed to determine if the individual has mastered the skills necessary for a certain purpose such as entry into the job market. There is also need for open-ended assessments such as observations, interviews, and manipulative tasks to assess skills which paper and pencil tests do not measure adequately.

Reports of pupil progress will surely be made. But while standardized tests will probably continue to dominate the testing scene for several years, there is an urgent need to begin reporting pupil progress in other terms, such as criterion tests and competency measures. This will also demand an immediate and extensive program of inservice education to instruct the general public on the meaning and interpretation of such data and to enable teachers to use testing as a vital part of the instructional process.

Large scale testing, whether involving all students or a random sample, can result in interpretations which have great influence on curriculum revisions and development. Test results can indicate, for example, that a particular mathematical topic is being taught at the wrong time in the student's development and that it might better be introduced later or earlier in the curriculum. Or, the results might indicate that students are confused about some topic as a result of inappropriate teaching procedures. In any case, test results should be carefully examined by educators with special skills in the area of curriculum development.

Conclusion

The present paper represents a preliminary attempt by the National Council of Supervisors of Mathematics to clarify and communicate its position on basic mathematical skills. The NCSM position establishes a framework within which decisions on program planning and implementation can be made. It also sets forth the underlying rationale for identifying and developing basic skills and for evaluating pupils' acquisition of these competencies. The NCSM position underscores the fundamental belief of the National Council of Supervisors of Mathematics that any effective program of basic mathematical skills must be directed not "back" but forward to the essential needs of adults in the present and future.

You are encouraged to make and distribute copies of this paper.

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